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METRIC EQUIVALENCE AS AN ALMOST SIMILARITY PROPERTY

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ABSTRACT

Various results that relate to almost similarity and other classes of operators such as isometry, normal, unitary and compact operators have been extensively discussed. It has been shown that if operators S and T are unitarily equivalent, then S is almost similar to T. Similarly, it has been shown that if operators A and B are such that A is almost similar to B and if A is Hermitian, then A and B are said to be unitarily equivalent. Metric equivalence property which is a new relation in operator theory has drawn much attention from mathematicians in the recent past. Two operators S and T are unitarily equivalent if they are metrically equivalent projections. It has been shown that if operators S and T are unitarily equivalent, then S is metrically equivalent to T. However, there is no literature that has been shown for the conditions under which metric equivalence and almost similarity coincide. In this paper we will therefore strive to establish the equivalence relation between metric equivalence property and almost similarity relation. To achieve this, properties of invertible operators, normal operators, similar operators, unitarily operators as well as projection and self-adjoint operators will be employed.

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1. INTRODUCTION

The class of almost similar operators was first introduced by (Jibril, 1996). He defined the class of almost similar operators as follows:

Two operators A and B are said to be *almost similar* if there exists an invertible operator N such that the following conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$
$$A^* + A = N^{-1}(B^* + B)N$$

(Jibril, 1996), proved various results that relate almost similarity and other classes of operators. (Musundi *et al.*, 2013) have shown that unitary equivalence of operators implies almost similarity of operators. Linear operators $T \in B(H)$ and $S \in B(K)$ are *unitarily equivalent* if there exist a unitary operator $U \in B(H,K)$ such that $UT = SU$ i.e.$T = U^*SU$, (Campbell & Gellar, 1977). (Nzimbi *et al.*, 2008) further studied the concept of almost similarity where they have shown that similarity implies almost similarity.

(Nzimbi *et al.*, 2013) introduced the concept of metric equivalence. They further proved that metric equivalence is an equivalence relation.

Two operators $A \in B(H)$and $B \in B(K)$ are said to be *metrically equivalent* if

$\|Ax\| = \|Bx\|$, (equivalently, $|<Ax, Ax>|^{\frac{1}{2}} = |<Bx, Bx>|^{\frac{1}{2}}$for all $x \in H$, that is $A^*A = B^*B$). Of great interest, (Nzimbi *et al.*, 2013) concretely discussed the spectral picture of metrically equivalent operators. They also gave some conditions under which metric equivalence of operators implies unitary equivalence of operators. Two operators A and B are said to be *similar* if there exist an invertible operator $N \in B(H,K)$ such that $NA = BN$or $A = N^{-1}BN$. If $S \in B(H)$ and $T \in B(H)$ are similar, then S^* and T^* are similar. It has been shown (Nzimbi *et al.*, 2008) that if $S \in B(H)$ and $T \in B(H)$are unitarily equivalent, then S and T are similar, If S and T are normal operators in a Hilbert space H, then S is unitarily equivalent to T if and only if S is similar to T. Thus it follows that two similar normal operators S and T are unitarily equivalent. (Musundi *et al.*, 2013) have showed that if operators S and T are unitary equivalent, then S is almost similar to T. Efforts towards establishment of the relation between the almost similarity and metric equivalence properties of operators could mark a significant contribution to the existing knowledge.

2. RELATED LITERATURE REVIEW

The review includes literature on some results on almost similarity, characterization of unitary equivalence of operators in terms of almost similarity. We also review metric equivalence relation and closely related relations on some classes of operators. In extension, we review some conditions under metric equivalence of operators implies unitary equivalence of operators.

2.1 Some Results on Almost Similarity

(Jibril, 1996) has shown that two operators A and B are said to be almost similar if there exist an invertible operator N such that the following two conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$

$A^* + A = N^{-1}(B^* + B)N.$

Theorem 2.1.1: (Nzimbi *et al.*, 2008)

Almost similarity of operators is an equivalence relation.

Proof: (Nzimbi *et al.*, 2008)

(i) Let $A \in B(H)$.Then $A^*A = N^{-1}(A^*A)N,$ where N is an invertible operator.

Also, $A^* + A = N^{-1}(A^* + A)N$. Hence $A^{a.s}A$.

(ii) Now suppose that $A^{a.s}B$, there exist an invertible operator N such that

$A^*A = N^{-1}(B^*B)N$ ……………...……………… (1)

And $A^* + A = N^{-1}(B^* + B)N$. ……………….. (2).

Since N is invertible, upon pre-multiplication of (1) and (2) by N and post multiplication of (1) and (2) by N^{-1} and applying the adjoint operation, we have

$B^*B = M^{-1}(A^*A)M,$ $B^* + B = M^{-1}(A^* + A)M$,where $N = M^{-1}$ which is an invertible operator, since N^{-1} is invertible. Hence $B^{a.s}A$.

(iii) Let A,B,C be in B(H).Suppose that $A^{a.s}B$and $B^{a.s}\,C$. Then we have

$A^*A = N^{-1}(B^*B)N,$ $A^* + A = N^{-1}(B^* + B)N$……………….. (3)

$B^*B = M^{-1}(C^*C)M,$ $B^* + B = M^{-1}(C^* + C)M$……………….. (4)

Where M and N are invertible operators. Using (3) and (4) we have that

$$A^*A = N^{-1}[M^{-1}(C^*C)M]N = (MN)^{-1}C^*C(MN) = S^{-1}(C^*C)S$$

$A^* + A = N^{-1}[M^{-1}(C^* + C)M]N = (MN)^{-1}C^* + C(MN) = S^{-1}(C^* + C)S$where $S = MN$, is invertible since M and N are invertible. It then follows that $A^{a.s}\,C$.

Theorem 2.1.2: (Campbell & Gellar, 1977)

An operator $T \in B(H)$ is Hermitian if and only if $(T + T^*)^2 \geq 4T^*T$

Theorem 2.2.2 helps us to prove the following results, where we assume the equality sign of this theorem.

Proposition 2.1.3: (Nzimbi *et al.*, 2008)

If $A, B \in B(H)$ such that $A^{a.s}B$ and B is Hermitian, then A is Hermitian.

Proof: (Nzimbi *et al.*, 2008)

Since $A^{a.s}B$ there exist an invertible operator N such that $A^*A = N^{-1}(B^*B)N$, on multiplying both sides by 4, we have,

$4A^*A = N^{-1}(4B^*B)N$ …………………….. (1)

Also $A^{a.s}B$, implies $A^* + A = N^{-1}(B^* + B)N$, on squaring both sides, we obtain,

$(A^* + A)^2 = N^{-1}(B^* + B)NN^{-1}(B^* + B)N$.

Thus

$N^{-1}(B^* + B)^2N = (A + A^*)^2$ ………………………… (2)

Since B is Hermitian, we have that $(B+B^*)^2 = (2B)^2 = 4B^2 = 4B^*B$. Substituting this in (2) we get

$(A + A^*)^2 = N^{-1}(4B^*B)N$ ……………… (3)

From (1) and (3) we have $4A^*A = (A + A^*)^2$ which shows that A is Hermitian, by

Theorem 2.1.2

Proposition 2.1.4: (Musundi *et al.*, 2013)

If $A, B \in B(H)$ such that A and B are unitarily equivalent, then $A^{a.s}B$.

Proposition 2.1.5: (Nzimbi *et al.*, 2008)

If $A, B \in B(H)$ such that $A^{a.s}B$, and if A is Hermitian, then A and B are unitarily equivalent.

Proof: (Nzimbi *et al.*, 2008)

By assumption, there exists an invertible operator N such that $A^* + A = N^{-1}(B^* + B)N$. Since A is Hermitian and $A^{a.s}B$ by proposition 2.1.3, B is Hermitian. Thus we have $2A = N^{-1}2BN$ which implies that $A = N^{-1}BN$. This implies that, A and B are similar and since both operators are normal (both A and B are Hermitian), they are unitarily equivalent.

Remark 2.1.6: (Nzimbi *et al.*, 2008)

The above proposition gives a condition under which almost similarity of operators implies similarity.

Theorem 2.1.7: Fuglede Commutativity Theorem(Rudin, 1991)

Assume that $A, B, T \in B(H)$, where A and B are normal, and AT=TB. Then $A^*T = TB^*$

Theorem 2.1.8: (Nzimbi *et al.*, 2008)

If $T \in B(H)$ is invertible, then T has a unique polar decomposition $T = UP$, with U an isometry (which is in fact a unitary) and $P \geq 0$. If $T \in B(H)$ is normal, then T has a polar decomposition $T = UP$ in which U and P commute with each other and T.

Theorem 2.1.9: (Nzimbi *et al.*, 2008)

Suppose $A, B, T \in B(H)$, A and B are normal, T is invertible, and $A = TBT^{-1}$. If $T = UP$ is the polar decomposition of T, then $A = UBU^{-1}$.

Remarks 2.1.10: (Nzimbi *et al.*, 2008)

This theorem asserts that similar normal operators are actually unitarily equivalent .The following results shows that unitary equivalence preserves normality of operators.

Theorem 2.1.11: (Sitati *et al.*, 2013)

If T is a normal operator and $S \in B(H)$ is unitarily equivalent to T, then S is normal.

Theorem 2.1.12: (Nzimbi*et al.*, 2013)

Two similar normal operators S and T are unitarily equivalent.

Remark 2.1.13: *Having looked at the properties of almost similarity operators, we will employ these properties and from the definition of a metric equivalence relation to establish conditions when almost similarity implies metric equivalence.*

2.2 Metric Equivalence of Some Operators

Recall that two operators $A \in B(H)$and $B \in B(K)$ are said to be *metrically equivalent* if $\|Ax\| = \|Bx\|$, (equivalently, $|< Ax, Ax >|^{\frac{1}{2}} = |< Bx, Bx >|^{\frac{1}{2}}$for all $x \in H$, that is

$A^*A = B^*B$)

The numerical range W(T) of an operator $T \in B(H)$ is defined as

$w(T) = \{\lambda \in \mathbb{C}: \lambda = < Tx, x >, \| x \| = 1\}$and

the numerical radius $r(T)$ of T is defined as $r(T) = \sup\{|\lambda| : \lambda \epsilon W(T)\}$.(Kubrusly,1997)

An operator T is said to be normaloid if $r(T) = \| T \|$, (equivalently, $\| T^n \| = \| T \|^n$). In complex Hilbert space H, every normal operator is normaloid and so is every positive operator.

Theorem 2.2.1: (Nzimbi *et al.*, 2013)

If T is a normal operator and $S \in B(H)$ is unitarily equivalent to T, then S is normal.

Theorem 2.2.2: (Dragomir, 2007)

A necessary and sufficient condition that an operator $T \in B(H)$be normal is that$\| Tx \| = \| T^*x \|$ for every $x \in H$.

Corollary 2.2.3: (Nzimbi *et al.*, 2013)

An operator $T \in B(H)$ is normal if and only if T and T^* are metrically equivalent.

Theorem 2.2.4: (Nzimbi et al., (2013)

If T is a normal operator, then there exist a unitary operator U such that $T^* = UT$.

3. MAIN RESULTS

3.1 Relationship between metrically equivalence operator and almost similarity operators.

To show this relationship we need the following results:

Theorem 3.1.1: (Nzimbi *et al.*, 2013)

Two operators $A \in B(H)$and $B \in B(K)$ are said to be *metrically equivalent* if

$\|Ax\| = \|Bx\|$, (equivalently, $|< Ax, Ax >|^{\frac{1}{2}} = |< Bx, Bx >|^{\frac{1}{2}}$for all $x \in H$, that is

$A^*A = B^*B$).

Corollary 3.1.2: (Nzimbi*et al.*, 2013)

If S and T are metrically equivalent normal operators, then there exist a unitary operator U such that $S = UT$.

Theorem 3.1.3: (Nzimbi*et al.*, (2013)

If T is a normal operator, then there exist a unitary operator U such that $T^* = UT$.

Corollary 3.1.4: (Nzimbi*et al.*, 2013)

An operator $T \in B(H)$ is normal if and only if T and T^* are metrically equivalent.

Theorem 3.1.5: (Jibril, 1996),

Two operators A and B are said to be *almost similar* if there exists an invertible operator N such that the following conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$
$$A^* + A = N^{-1}(B^* + B)N$$

Proposition 3.1.6: (Nzimbi*et al.*, 2008)

If $T, S \in B(H)$ such that $T^{a.s}S$ and S is Hermitian, then T is Hermitian.

Corollary 3.1.7: All hermitian operators are normal.

Corollary 3.1.8: (Nzimbi *et al.*, 2013)

An operator $T \in B(H)$ is normal if and only if T and T^* are metrically equivalent.

Definition 3.1.9: An operator P is said to be a projection operator if $P = P^*$ and $P = P^2$

Theorem 3.1.10: (Dragomir, 2007)

A necessary and sufficient condition that an operator $S \in B(H)$ be normal is that $\| Sx \| = \| S^*x \|$ for every $x \in H$

Thus our main results follows:

Theorem 3.2.1: *If $S, T \in B(H)$ such that $S^{a.s}T$, then S and T are metrically equivalent.*

Proof

Suppose $S, T \in B(H)$ *such that* $S^{a.s}T$ and let S be hermitian. An operator S is said to be hermitian if $S = S^*$

We note that S and T are said to be *almost similar* if there exists an invertible operator N such that the following conditions are satisfied:

$$S^*S = N^{-1}(T^*T)N$$
$$S^* + S = N^{-1}(T^* + T)N$$

Assume that there exist an invertible operator N such that

$S^* + S = N^{-1}(T^* + T)N$. Since S is hermitian and $S^{a.s}T$, then byProposition 4.1.6 T is hermitian Thus we have

$2S = N^{-1}2TN \Rightarrow S = N^{-1}TN$. So that S and T are similar and being hermitian operators it means they are also normal, so they are unitarily equivalent. i.e.$S = U^*TU$.

Now if $S = U^*TU$ then $S^* = U^*T^*U$

Let $x \in H$ and since S is normal then by Theorem 4.1.10,$\| Sx \| = \| S^*x \|$

By the definition of a norm

$\| Sx \|^2 = < Sx, Sx >$

$= < S^*Sx, x >$

$= < U^*T^*UU^*TUx, x >$

$= < U^*T^*TUx, x >$ by theorem 4.1.3 we get

$= < U^*UTT^*U^*Ux, x >$

$= < TT^*x, x >$ but since T is normal then $TT^* = T^*T$ then, we have

$= < T^*Tx, x >$

$= < Tx, Tx > = \| Tx \|^2$ so that

$\| Sx \| = \| Tx \|$

Thus S and T are metrically equivalent.

Theorem 3.2.2: *If T and S are metrically equivalent projections operators, then T and S are almost similar.*

Proof

Suppose that T and S are metrically equivalent projections operators. We recall that an operator T is said to be a projection operator if $T = T^*$ and $T = T^2$

Since T and S are projections ($T = T^*$ and $T = T^2, S = S^*$ and $S = S^2$) then they are self-adjoint and by Corollary 4.1.7 it implies that they are normal operators.

Since T and S are normal operators then they exist a unitary operator U such that $T^* = UT$.

Thus S being a projection operator, self-adjoint and also metrically equivalent to T, we have

$S = S^2 = S^*S = T^*T = UTT^*U^* = UT^2U^* = UTU^*$ i.e. $S = UTU^*$, which shows that S and T are unitarily equivalent.

Again since $S = UTU^*$ then $S^* = U^*T^*U$

and thus

$S^*S = U^*T^*UU^*TU = U^*T^*TU = U^{-1}T^*TU$ ………………..…………….(i)

$S^* + S = U^*T^*U + U^*TU = U^*(T^* + T)U = U^{-1}(T^* + T)U$ ……………….(ii)

Equation (i) and (ii) shows that $S^{a.s}T$.

Remark. *For two operators that are metrically equivalent to imply almost similarity, then the two operators must be projection operators. This gives the condition under which metric equivalence implies almost similarity.*

Corollary 3.2.3: *Two metrically equivalent operators implies almost similarity if and only if they are projection operators.*

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