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Implicative algebras and Heyting algebras can be residuated lattices

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Abstract

The commutative residuated lattices were first introduced by M. Ward and R.P. Dilworth as generalization of ideal lattices of rings. Complete studies on residuated lattices were developed by H. Ono, T. Kowalski, P. Jipsen and C. Tsinakis. Also, the concept of lattice implication algebra is due to Y. Xu. And Luitzen Brouwer founded the mathematical philosophy of intuitionism, which believed that a statement could only be demonstrated by direct proof. Arend Heyting, a student of Brouwer's, formalized this thinking into his namesake algebras. In this paper, we investigate the relationship between implicative algebras, Heyting algebras and residuated lattices. In fact, we show that implicative algebras and Heyting algebras can be described as residuated lattices.

Keywords: Residuated lattice; implicative algebra; Heyting algebra.

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The concept of a commutative residuated lattice was firstly introduced by M. Ward and R. P. Dilworth [5] as a generalization of ideal lattices of rings. In their original definition a residuated lattice was what we would call an integral commutative one. Basic properties and facts in this restricted setting were known, but in [1] one can find the first thorough study of residuated lattices in their generality.

For a survey of residuated lattices we refer to [2].

In [6] Xu introduced the concept of lattice implication algebra and quasi implication algebra as a bounded lattice satisfying a system of axioms and studied certain properties. Later many authors like Jun et al. [3] have studied the properties of filters and fuzzy filters of lattice implication algebras and quasi lattice implication algebras. Also Zhu Yiquan and Tu have introduced an equivalent definition for lattice implication algebra in [7]. In [4] Kolluru and Bekele gave an equivalent definition of implication lattice of Xu by simplifying the axioms of his definition and called implicative algebra.

In this paper we shown that implicative algebras and Heyting algebras can be residuated lattices.

Definition 1 . *Let L be a non empty set, $1 \in L$ and $\odot$ be a partial binary operation on L. Then $(L, \odot, 1)$ is called* ***a partial monoid*** *if the following holds:*

(i) $x \odot (y \odot z) = (x \odot y) \odot z \qquad \forall x, y, z \in L$

(ii) $x \odot 1 = x = 1 \odot x \qquad \forall x \in L.$

Definition 2 . ***A commutative residuated lattice*** *is an algebra $\mathfrak{L} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of the type (2, 2, 2, 2, 0, 0) satisfying the following conditions:*

(L_1) *$(L, \vee, \wedge, 0, 1)$ is a bounded lattice;*

(L_2) *$(L, \odot, 1)$ is a commutative monoid;*

(L_3) *The adjointness condition $x \odot y \leq z$ iff $x \leq y \rightarrow z \qquad$ for all $x, y, z \in L$.*

Further, we say that $\mathfrak{L}$ is a DN- residuated lattice if it satisfies the double negation law i.e. $\neg\neg x = x$ for all $x \in L$.

Corollary 1 . *For each residuated lattice $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ and for all $a, b, c \in L$ we have:*

(i) $a \leq b \Leftrightarrow a \rightarrow b = 1$;

(ii) $a \odot a \rightarrow b \leq b$ and $b \leq a \rightarrow a \odot b$;

(iii) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$.

Proof.

(i) We have $a \leq b$ iff $a \odot 1 \leq b$, and hence iff $1 = a \rightarrow b$ by the adjointness condition.

(ii) Form the reflexivity of the lattice ordering $\leq$ and by using the adjointness condition we can get $a \odot (a \rightarrow b) \leq b$ and $b \leq a \rightarrow (a \odot b)$.

(iii) By the monotonicity of $\odot$ we have $a \odot b \leq a \odot (b \vee c)$ as well as $a \odot c \leq a \odot (b \vee c)$ and thus $(a \odot b) \vee (a \odot c) \leq a \odot (b \vee c)$.
And from $b \leq a \rightarrow (a \odot b)$ and the monotonicity of $\rightarrow$ in the second argument we get

$$b \leq a \rightarrow ((a \odot b) \vee (a \odot c)).$$

Similarly, we get

$$c \leq a \rightarrow ((a \odot b) \vee (a \odot c)).$$

And hence

$$(b \vee c) \leq a \rightarrow ((a \odot b) \vee (a \odot c)).$$

Thus by the adjointness condition gives also

$$a \odot (b \vee c) \leq (a \odot b) \vee (a \odot c).$$

■

Definition 3 . *An algebra $\mathfrak{I} = (I, \rightarrow, \neg, 0, 1)$ of the type $(2, 1, 0, 0)$ called* ***implicative algebra*** *if it satisfies the following equations:*

(I_1) $x \to (y \to z) = y \to (x \to z)$;

(I_2) $1 \to x = x$;

(I_3) $x \to 1 = 1$;

(I_4) $x \to y = \neg y \to \neg x$;

(I_5) $(x \to y) \to y = (y \to x) \to x$;
where $(\neg 0 = 1)$

Define a relation $\leq$ on an implicative algebra $\mathfrak{I}$ as follows:

$$x \leq y \Leftrightarrow x \to y = 1.$$

Also, we define two binary operation $\vee$ and $\wedge$ on I by

$$x \vee y = (x \to y) \to y = (y \to x) \to x$$
$$x \wedge y = \neg((y \to x) \to \neg y) = \neg((x \to y) \to \neg x)$$

Lemma 1 . *[4] In any implicative algebra $\mathfrak{I}$, the following hold:*

(1) $(I, \leq, 0, 1)$ *is a bounded poset;*

(2) $x \wedge y \leq x, y \leq x \vee y$;

(3) $x \vee y$ *is the least upper bound of* x, y;

(4) $x \wedge y$ *is the greatest upper bound of* x, y.

Theorem 1 . *Let $\mathfrak{I} = (I, \to, \neg, 0, 1)$ be an implicative algebra with $\leq$ its induced order. Define $x \odot y = \neg(x \to \neg y)$. Then $\mathfrak{R}(I) = (I, \leq, \odot, 0, 1)$ is a commutative residuated lattice.*

Proof.

(L_1) $(I, \vee, \wedge, 0, 1)$ is a lattice by Lemma 1.

(L_2)

$$\begin{aligned} x \odot (y \odot z) &= \neg(x \to \neg(y \odot z)) \\ &= (x \to (y \to \neg z)) \\ &= \neg(x \to (z \to \neg y)) \\ &= \neg(z \to (x \to \neg y)) \\ &= \neg(\neg(x \to \neg y) \to \neg z) \\ &= \neg((x \odot y) \to \neg z) \\ &= (x \odot y) \odot z. \end{aligned}$$

$x \odot 1 = \neg(x \to \neg 1) = \neg(x \to 0) = \neg\neg x = x$. And $1 \odot x = \neg(1 \to \neg x) = \neg\neg x = x$. Also, $x \odot y = \neg(x \to \neg y) = \neg(y \to \neg x) = y \odot x$.

(H_1) *$(H, \vee, \wedge, 0, 1)$ is a distributive lattice;*

(H_2) $x \wedge 0 = 0, x \vee 1 = 1$;

(H_3) $x \to x = 1$;

(H_4) $(x \to y) \wedge y = y,\ x \wedge (x \to y) = x \wedge y$;

(H_5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z), (x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Theorem 3 . *Let $\mathfrak{H} = (H, \vee, \wedge, \to, 0, 1)$ be a heyting algebra with $x \odot y = x \wedge y$. Then $\Re(H) = (H, \leq, \odot, 0, 1)$ is a commutative residuated lattice.*
Proof.

(L_1) $(H, \vee, \wedge, 0, 1)$ is a bounded lattice by (H_1).

(L_2) $(H, \odot, 1)$ is a commutative monoid
$x \odot (y \odot z) = x \wedge (y \wedge z) = (x \wedge y) \wedge z = (x \odot y) \odot z$,
$x \odot 1 = x \wedge 1 = x = 1 \wedge x = 1 \odot x$,
$x \odot y = x \wedge y = y \wedge x = y \odot x$.

(L_3) If $x \leq y \to z$ then $x \odot y \leq (y \to z) \odot y = y \odot (y \to z)$ by (L_2).
This implies that $x \wedge y \leq y \wedge (y \to z) \leq y \wedge z$ by (H_4). So, we get $x \odot y \leq z$.
Conversely, if $x \odot y \leq z$, then $x \wedge y \leq z$. And
$y \to (x \wedge y) \leq y \to z$ by the monotonicity of $\to$ in the second argument.
$(y \to x) \wedge (y \to y) \leq y \to z$ by (H_5)
$(y \to x) \wedge 1 \leq y \to z$ by (H_3)
$y \to x \leq y \to z$
$(y \to x) \wedge x \leq (y \to z) \wedge x \leq y \to z$
$x \leq y \to z$ by (H_4).

■

Theorem 4 . *Let $\mathfrak{L} = (L, \leq, \odot, 0, 1)$ be a commutative residuated lattice with $x \wedge y = x \odot y$, $x \vee y = \neg x \to y$. Then $\mathfrak{H}(L) = (L, \vee, \wedge, \to, 0, 1)$ is a heyting algebra.*
Proof.

(H_1) According to Corollary 1 (iii) and $x \wedge y = x \odot y$.

(H_2) $x \wedge 0 = 0, x \vee 1 = 1$ by (L_1).

(H_3) $x \to x = \neg x \vee x = 1$ by (L_1).

(H_4) $(x \to y) \wedge y = (\neg x \vee y) \wedge y = y$ and

$$\begin{aligned} x \wedge (x \to y) &= x \wedge (\neg x \vee y) \\ &= (x \wedge \neg x) \vee (x \wedge y) \qquad by(H_1) \\ &= x \wedge y. \end{aligned}$$

(H_5)

$$\begin{aligned} x \to (y \wedge z) &= \neg x \vee (y \wedge z) \\ &= (\neg x \vee y) \wedge (\neg x \vee z) \qquad by(H_1) \\ &= (x \to y) \wedge (x \to z). \end{aligned}$$

And $(x \vee y) \to z = \neg(x \vee y) \vee z = (\neg x \vee z) \wedge (\neg y \vee z) = (x \to z) \wedge (y \to z)$.

■

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