



## Multiple solutions for a kind of periodic boundary value problems via variation approach <sup>12</sup>

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### Abstract

In this paper, the existence of at least two solutions for periodic boundary value problems is obtained by the critical point theory. The interest is that the nonlinear term includes the first-order derivative and may not satisfy the classical Ambrosetti-Rabinowitz condition.

**Keywords:**  $p$ -laplacian; boundary value problem; variational; periodic solution.

**Mathematics Subject Classification 2000:** 34B24.

### 1. Introduction

This paper considers the existence of at least two periodic solutions for the following kind of boundary value problems (for short BVPs)

$$\begin{cases} (\phi_p(x'(t)))' = (a(t)\phi_p(x(t)) + f(t, x(t)))g(x'(t)), & t \in [0, T] \setminus \{t_1, \dots, t_k\} \\ \Delta G(x'(t_i)) = I_i(x(t_i)), & i = 1, \dots, k \\ x(0) - x(T) = 0, & x'(0) - x'(T) = 0. \end{cases} \quad (1.1)$$

Here,  $p > 1$ ,  $T > 0$ ,  $\phi_p(x) = |x|^{p-2}x$ ,  $G(x) = \int_0^x \frac{(p-1)|s|^{p-2}}{g(s)} ds$ .

The periodic BVPs have received a lot of attention. Many works have been carried out to discuss the existence of at least one solution, multiple solutions. The methods therein mainly depend on lower and upper solutions with monotone iterative<sup>[1]</sup> and fixed point theorems<sup>[2]</sup>, etc. Recently, variational methods have been used to study the existence of periodic solutions, such as [3,4]. Moreover, Ambrosetti and Rabinowitz<sup>[10]</sup> established the existence of nontrivial solutions for Dirichlet problems under the well known Ambrosetti-Rabinowitz condition: there exist some  $\mu > 2$  and  $R > 0$  such that

$$0 < \mu \int_0^x f(t, s) ds \leq f(t, x)x$$

for all  $t \in [0, T]$  and  $|x| \geq R$ . Since then, the AR-condition has been used extensively. By the usual AR-condition, it is easy to show that the Euler-Lagrange functional associated with the system has the mountain pass geometry and the Palais-Smale sequence is bounded.

Since it is not easy to verify the corresponding Euler functional satisfying (PS)-condition, few papers consider the boundary value problems with the nonlinear term including the first-order derivative. G. A. Afrouzi and S. Heidarkhani<sup>[8]</sup> proved the existence of at least three weak solutions for the Dirichlet problem

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$$\begin{cases} y'' + \lambda h(y')f(t, y) = 0, & a < t < b \\ y(a) = y(b) = 0, \end{cases}$$

where  $a, b \in \mathbb{R}$ ,  $\lambda > 0$ . Based on the three critical points theorem in [5], the authors did not need to verify (PS)-condition of the corresponding Euler functional.

To the best of our knowledge, no people consider the existence of at least two solutions for periodic BVP with the nonlinearity including  $x'$  by using variational methods. As a result, the goal of this paper is to fill the gap in this area.

The outline of the paper is as follows. In the forthcoming section, we present some general results. In section 3, we exhibit the existence of at least two solutions. Throughout, assume  $a(t) \in C([0, T]; (0, +\infty))$ ,  $F(t, x) = \int_0^x f(t, u)du$ ,  $G(x) = \int_0^x \frac{(p-1)|s|^{p-2}}{g(s)} ds$  and

$(A_1)f(t, x): [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

$(A_2)g(x): \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there exist constants  $M > m > 0$  such that  $M \geq g(x) \geq m$ ,  $x \in \mathbb{R}$ .

$(A_3)I_i(x): \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $i = 1, \dots, k$ .

**Remark 1.1** From the expression of  $G(x)$ , one has  $\frac{d}{dt}G(x) = \frac{(p-1)|x|^{p-2} dx}{g(x) dt} = \frac{\frac{d}{dt}(\phi_p(x(t)))}{g(x)}$ ,  $\frac{1}{M}|x|^p \leq G(x)x \leq \frac{1}{m}|x|^p$ ,  $\frac{1}{M_p}|x|^p \leq \int_0^x G(s)ds \leq \frac{1}{m_p}|x|^p$ .

## 2. Preliminary

The Sobolev space  $W^{1,p}[0, T]$  is defined by

$$W^{1,p}[0, T] = \{x: [0, T] \rightarrow \mathbb{R} | x \text{ is absolutely continuous, } x' \in L^p(0, T; \mathbb{R}), x(0) = x(T)\} \quad (2.1)$$

and is endowed with the norm

$$\|x\| = \left( \int_0^T |x(t)|^p dt + \int_0^T |x'(t)|^p dt \right)^{\frac{1}{p}}. \quad (2.2)$$

Then, from [3],  $W^{1,p}[0, T]$  is a separable and reflexive Banach space. Next, we show some basic knowledge.

**Theorem 2.1**<sup>[6]</sup> For the functional  $F: M \subset X \rightarrow [-\infty, +\infty]$  with  $M \neq \emptyset$ ,  $\min_{u \in M} F(u) = \alpha$  has a solution in case the following holds:

- (i)  $X$  is a real reflexive Banach space;
- (ii)  $M$  is bounded and weak sequentially closed, i.e., for each sequence  $\{u_n\}$  in  $M$  such that  $u_n \rightharpoonup u$  as  $n \rightarrow \infty$ , we always have  $u \in M$ ;
- (iii)  $F$  is weakly sequentially lower semi-continuous on  $M$ .

**Theorem 2.2**<sup>[3]</sup> Let  $E$  be a Banach space and  $\varphi \in C^1(E, \mathbb{R})$  satisfy (PS)-condition. Assume there exist  $x_0, x_1 \in E$ , and a bounded open neighborhood  $\Omega$  of  $x_0$  such that  $x_1 \notin \Omega$  and

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x \in \partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h: [0, 1] \rightarrow E \text{ is continuous and } h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0, 1]} \varphi(h(s)).$$

Then,  $c$  is a critical value of  $\varphi$ , that is, there exists  $x^* \in E$  such that  $\varphi'(x^*) = 0$  and  $\varphi(x^*) = c$ , where  $c > \max\{\varphi(x_0), \varphi(x_1)\}$ .

For  $x \in C[0, T]$ , suppose  $\|x\|_\infty = \max_{t \in [0, T]} |x(t)|$ ,  $\|x\|_m = \min_{t \in [0, T]} |x(t)|$ .

**Lemma 2.1** If  $x \in W^{1,p}[0, T]$ , then,  $\|x\|_\infty \leq C \|x\|$  where  $C = T^{\frac{1}{p}} + T^{\frac{1}{q}}$ .

**Lemma 2.2**<sup>[7]</sup> There exists a positive constant  $c_p$  such that

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq \begin{cases} c_p|x - y|^p, & p \geq 2, (2.3) \\ c_p \frac{|x - y|^2}{(|x| + |y|)^{2-p}}, & 1 < p < 2(2.4) \end{cases} \quad (2.3)$$

for any  $x, y \in \mathbb{R}^N$ ,  $|x| + |y| \neq 0$ . Here,  $(x, y) = x \cdot y^T$ .

In the following, we state the (C) condition<sup>[8]</sup>. Let  $\varphi$  be a Frechet differential functional and  $H$  be a Banach space,

(C) Every sequence  $(x_n)_{n \in \mathbb{N}} \subset H$  such that the following conditions hold:

(i)  $(\varphi(x_n))_{n \in \mathbb{N}}$  is bounded,

(ii)  $(1 + \|x_n\|_H) \|\varphi'(x_n)\|_{H^*} \rightarrow 0$  as  $n \rightarrow \infty$

has a subsequence which converges strongly in  $H$ .

### 3. Existence of at least two solutions

Define

$$\varphi(x) = \int_0^T [F(t, x) + \frac{1}{p}a(t)|x(t)|^p + \int_0^{x'} G(u)du]dt + \sum_{i=1}^k \int_0^{x(t_i)} I_i(t)dt, \quad (3.1)$$

Obviously,  $\varphi$  is continuously differentiable on  $W^{1,p}[0, T]$  and by computation, one has

$$\langle \varphi'(x), y \rangle \geq \int_0^T f(t, x)y dt + \int_0^T a(t)\phi_p(x)y dt + \int_0^T G(x')y' dt + \sum_{i=1}^k I_i(x(t_i))y(t_i), \quad x, y \in W^{1,p}[0, T]. (3.2)$$

**Lemma 3.1** If  $x \in W_T^{1,p}$  is a critical point of  $\varphi$ , then,  $x$  is a solution to BVP(1.1).

**Lemma 3.2** Assume  $(A_1)$ ,  $(A_2)$ , and the following conditions

$(B_1)$  there exists some constants  $\mu > p$  and  $\gamma_1, \gamma_2$  satisfying

$$\mu F(t, x) - x f(t, x) \geq \gamma_1 x^p - \gamma_2, \quad x \in \mathbb{R};$$

$(B_2) \lim_{x \rightarrow +\infty} \frac{f(t, x)}{|x|^{p-1}} = -\infty, \lim_{x \rightarrow -\infty} \frac{f(t, x)}{|x|^{p-1}} = +\infty, t \in [0, T];$

$$(B_3) \mu \min\{\|a\|_m, \frac{1}{M}\} > p \max\{\|a\|_\infty, \frac{1}{m}\}$$

$$(B_4) 0 \leq I_i(x) \cdot x, I_i(x) \leq b_i + c_i x^{\tau-1}, \tau < p, b_i, c_i \in \mathbb{R}^+, i = 1, \dots, k$$

hold, then the functional  $\varphi$  satisfies (C)-condition.

**Proof.** Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $W^{1,p}[0, T]$  such that  $\varphi(x_n)$  is bounded and  $\|\varphi'(x_n)\| \times (1 + \|x_n\|) \rightarrow 0$  as  $n \rightarrow \infty$ . Then, there exist a constant  $C_1 > 0$  and a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}, \varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  such that

$$|\varphi(x_n)| \leq C_1, \quad |\langle \varphi'(x_n), x_n \rangle| \leq \varepsilon_n. \quad (3.3)$$

Suppose  $\|x_n\| \rightarrow \infty, n \rightarrow \infty$ . Set  $y_n = \frac{x_n}{\|x_n\|}$  for all  $n \geq 1$ . Obviously,  $\|y_n\| = 1$ , that is,  $(y_n)_{n \in \mathbb{N}}$  is a bounded sequence in  $W^{1,p}[0, T]$ . Going to a subsequence if necessary, we may assume that

$$y_n \rightharpoonup y \text{ in } W^{1,p}[0, T], \quad y_n(t) \rightarrow y(t) \text{ in } [0, T].$$

From (3.3),

$$\left| \frac{\int_0^T a(t)|x_n|^p dt + \int_0^T G(x_n')x_n' dt}{\|x_n\|^p} + \int_0^T \frac{f(t, x_n)y_n}{\|x_n\|^{p-1}} dt + \frac{\sum_{i=1}^k I_i(x_n(t_i))x_n(t_i)}{\|x_n\|^p} \right| \leq \frac{\varepsilon_n}{\|x_n\|^p}. \quad (3.4)$$

Moreover,

$$\min\{\frac{1}{M}, \|a\|_m\} \leq \frac{\int_0^T a(t)|x_n|^p dt + \int_0^T G(x_n')x_n' dt}{\|x_n\|^p} \leq \max\{\frac{1}{m}, \|a\|_\infty\}.(3.5)$$

Let  $\Omega_0^+ = \{t \in [0, T], y(t) > 0\}$ ,  $\Omega_0^- = \{t \in [0, T], y(t) < 0\}$ , then,  $x_n(t) \rightarrow +\infty$  for  $t \in \Omega_0^+$ ,  $x_n(t) \rightarrow -\infty$  for  $t \in \Omega_0^-$ ,  $n \rightarrow \infty$ . By the hypothesis,

$$\frac{f(t, x_n(t))}{(x_n(t))^{p-1}} \rightarrow -\infty, \quad t \in \Omega_0^+, \quad n \rightarrow \infty.$$

If  $meas\Omega_0^+ > 0$ , then,

$$(y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} \rightarrow -\infty, \quad t \in \Omega_0^+, \quad n \rightarrow \infty.$$

If  $meas\Omega_0^- > 0$ , then,

$$(y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} = y_n(t)|y_n(t)|^{p-1} \frac{f(t, x_n(t))}{|x_n(t)|^{p-1}} \rightarrow -\infty, \quad t \in \Omega_0^-, \quad n \rightarrow \infty.$$

Moreover,

$$\begin{aligned} \int_0^T \frac{f(t, x_n)y_n}{\|x_n\|^{p-1}} dt &= \int_{\Omega_0^+} (y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} dt + \int_{\Omega_0^-} (y_n(t))^p \frac{f(t, x_n(t))}{(x_n(t))^{p-1}} dt \\ &\quad + \int_{[0, T] \setminus (\Omega_0^+ \cup \Omega_0^-)} \frac{f(t, x_n(t))x_n(t)}{\|x_n\|^p} dt. \end{aligned}$$

Then, from  $(B_2)$ , there exists some constant  $k > 0$ ,  $f(t, x)x < 0$  for  $|x| > k$ . For  $|x| \leq k$ ,  $f(t, x)$  is bounded by continuity, hence,  $\int_{[0, T] \setminus (\Omega_0^+ \cup \Omega_0^-)} \frac{f(t, x_n(t))x_n(t)}{\|x_n\|^p} dt$  is bounded or negative. By  $(B_3)$  and the discussion above, one has

$$\int_0^T \frac{f(t, x_n)y_n}{\|x_n\|^{p-1}} dt \rightarrow -\infty, \quad n \rightarrow \infty.$$

This reaches a contradiction, that is,  $meas\Omega_0^+ = meas\Omega_0^- = 0$ . Then, one can conclude that  $y(t) = 0$  for a.e.  $t \in [0, T]$ . Hence,  $y(t) \equiv 0$ ,  $t \in [0, T]$ .

Moreover,

$$\begin{aligned} \frac{1}{p} \min\{\|a\|_m, \frac{1}{M}\} \|x_n\|^p &\leq \int_0^T [\frac{1}{p} a(t)|x_n(t)|^p + \int_0^{x_n'} G(u)du] dt \\ &\leq \frac{1}{p} \max\{\|a\|_\infty, \frac{1}{m}\} \|x_n\|^p, \end{aligned}$$

$$\min\{\|a\|_m, \frac{1}{M}\} \|x_n\|^p \leq \int_0^T [a(t)|x_n(t)|^p + G(x_n')x_n'] dt \leq \max\{\|a\|_\infty, \frac{1}{m}\} \|x_n\|^p.$$

$$0 \leq I_i(x_n(t_i)) \cdot x_n(t_i) \leq b_i|x_n(t_i)| + c_i|x_n^\tau(t_i)| \leq cb_i \|x_n\| + c^\tau c_i \|x_n\|^\tau$$

$$|\int_0^{x_n(t_i)} I_i(t) dt| = |I_i(\zeta) \cdot x_n(t_i)| \leq b_i|x_n(t_i)| + c_i|x_n^\tau(t_i)| \leq cb_i \|x_n\| + c^\tau c_i \|x_n\|^\tau$$

Here  $c = T^{-\frac{1}{p}} + T^{\frac{1}{q}}$ . Hence,

$$-C_1 - \frac{1}{p} \max\{\|a\|_\infty, \frac{1}{m}\} \|x_n\|^p - \sum_{i=1}^k (cb_i \|x_n\| + c^\tau c_i \|x_n\|^\tau)$$

$$\leq \int_0^T F(t, x_n) dt \leq C_1 - \frac{1}{p} \min\{ \| a \|_m, \frac{1}{M} \} \| x_n \| ^p,$$

$$\begin{aligned} & -\varepsilon_n + \min\{ \| a \|_m, \frac{1}{M} \} \| x_n \| ^p \\ & \leq - \int_0^T f(t, x_n) x_n dt \\ & \leq \varepsilon_n + \max\{ \| a \|_\infty, \frac{1}{m} \} \| x_n \| ^p + \sum_{i=1}^k (cb_i \| x_n \| + c^\tau c_i \| x_n \|^\tau), \end{aligned}$$

Then, one has

$$\begin{aligned} & -\frac{\mu C_1}{\| x_n \| ^p} - \frac{\mu}{p} \max\{ \| a \|_\infty, \frac{1}{m} \} \\ & \leq \int_0^T \frac{\mu F(t, x_n)}{\| x_n \| ^p} dt \leq -\frac{\mu}{p} \min\{ \| a \|_m, \frac{1}{M} \} + \frac{\mu C_1}{\| x_n \| ^p} - \sum_{i=1}^k (cb_i \frac{1}{\| x_n \|^{p-1}} - c^\tau c_i \frac{1}{\| x_n \|^{p-\tau}}), \\ & -\frac{\varepsilon_n}{\| x_n \| ^p} + \min\{ \| a \|_m, \frac{1}{M} \} \\ & \leq - \int_0^T \frac{f(t, x_n) x_n}{\| x_n \| ^p} dt \leq \max\{ \| a \|_\infty, \frac{1}{m} \} + \frac{\varepsilon_n}{\| x_n \| ^p} + \sum_{i=1}^k (cb_i \frac{1}{\| x_n \|^{p-1}} + c^\tau c_i \frac{1}{\| x_n \|^{p-\tau}}). \end{aligned}$$

Moreover,

$$\begin{aligned} & -\frac{\mu}{p} \max\{ \| a \|_\infty, \frac{1}{m} \} + \min\{ \| a \|_m, \frac{1}{M} \} \\ & \leq \lim_{n \rightarrow +\infty} \int_0^T \frac{\mu F(t, x_n) - x_n f(t, x_n)}{|x_n|^p} |y_n|^p dt \\ & \leq -\frac{\mu}{p} \min\{ \| a \|_m, \frac{1}{M} \} + \max\{ \| a \|_\infty, \frac{1}{m} \} < 0. \end{aligned} \tag{3.6}$$

By (B<sub>1</sub>),

$$\lim_{n \rightarrow +\infty} \frac{\mu F(t, x_n) - x_n f(t, x_n)}{|x_n|^p} |y_n|^p \geq \lim_{n \rightarrow +\infty} \frac{\gamma_1 x_n^p - \gamma_2}{|x_n|^p} |y_n|^p \rightarrow 0,$$

which is contradictive with (3.6). Hence,  $(x_n)_{n \in N}$  is bounded in  $W^{1,p}[0, T]$ . By the compactness of the embedding  $W^{1,p}[0, T] \hookrightarrow C[0, T]$ , the sequence  $(x_n)_{n \in N}$  has a subsequence, again denoted by  $(x_n)_{n \in N}$  for convenience, such that

$$\begin{aligned} x_n & \rightharpoonup x \text{ weakly in } W^{1,p}[0, T], \\ x_n & \rightarrow x \text{ strongly in } C[0, T]. \end{aligned}$$

From (3.2), for  $m, n \in N$ ,

$$\begin{aligned} & \langle \varphi'(x_n) - \varphi'(x_m), x_n - x_m \rangle \\ & = \int_0^1 (f(t, x_n) - f(t, x_m))(x_n - x_m) dt + \int_0^1 a(t)(\phi_p(x_n) - \phi_p(x_m)) \end{aligned}$$

$$(x_n - x_m)dt + \int_0^1 (G(x'_n) - G(x'_m))(x'_n - x'_m)dt$$

$$+ \sum_{i=1}^k (I_i(x_n(t_i)) - I_i(x_m(t_i)))(x_n(t_i) - x_m(t_i)).$$

Since  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $C[0, T]$ ,  $f(t, x)$  is continuous in  $x$ ,  $G(x)$  is continuous, and  $|\langle \varphi'(x_n) - \varphi'(x_m), x_n - x_m \rangle| \leq (\|\varphi'(x_n)\| + \|\varphi'(x_m)\|) \|x_n - x_m\|$ ,  $\|x_n - x_m\|$  is bounded in  $W^{1,p}[0, T]$ ,  $\|\varphi'(x_n)\| + \|\varphi'(x_m)\| \rightarrow 0$ ,  $n, m \rightarrow \infty$ , one has  $\langle \varphi'(x_n) - \varphi'(x_m), x_n - x_m \rangle \rightarrow 0$ . Hence

$$\int_0^T (G(x'_n) - G(x'_m))(x'_n - x'_m)dt \rightarrow 0, \quad n, m \rightarrow \infty.$$

Moreover,

$$\int_0^T (G(x'_n) - G(x'_m))(x'_n - x'_m)dt = \int_0^T \left( \int_{x'_m}^{x'_n} \frac{(p-1)|s|^{p-2}}{g(s)} ds \right) (x'_n - x'_m)dt$$

$$\geq \frac{1}{M} \int_0^1 (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m)dt$$

$$\geq 0$$

then,

$$\int_0^T (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m)dt \rightarrow 0, \quad \text{as } n, m \rightarrow \infty. \quad (3.7)$$

If  $p \geq 2$ , from Lemma 2.2, there exists a positive constant  $c_p$  such that

$$\int_0^T (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m)dt \geq c_p \int_0^T |x'_n - x'_m|^p dt. \quad (3.8)$$

If  $p < 2$ , by Lemma 2.2, Hölder inequality and the boundedness of  $(x_n)_{n \in \mathbb{N}}$  in  $W^{1,p}$ , one has

$$\int_0^T |x'_n - x'_m|^p dt = \int_0^T \frac{|x'_n - x'_m|^p}{(|x'_n| + |x'_m|)^{\frac{p(2-p)}{2}}} (|x'_n| + |x'_m|)^{\frac{p(2-p)}{2}} dt$$

$$\leq \left( \int_0^T \frac{|x'_n - x'_m|^2}{(|x'_n| + |x'_m|)^{2-p}} dt \right)^{\frac{p}{2}} \left( \int_0^T (|x'_n| + |x'_m|)^p dt \right)^{\frac{2-p}{2}}$$

$$\leq c_p^{\frac{-p}{2}} \left( \int_0^T (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m) dt \right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}}$$

$$\left( \int_0^T (|x'_n|^p + |x'_m|^p) dt \right)^{\frac{2-p}{2}}$$

$$\leq c_p^{\frac{-p}{2}} \left( \int_0^T (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m) dt \right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}}$$

$$(\|x_n\|^p + \|x_m\|^p)^{\frac{2-p}{2}}. \quad (3.9)$$

From (3.7)-(3.9), we have  $\int_0^T |x'_n - x'_m|^p dt \rightarrow 0$  as  $n, m \rightarrow \infty$ . Then,  $\|x_n - x_m\| \rightarrow 0$ , that is,  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $W^{1,p}[0, T]$ . By the completeness of  $W^{1,p}[0, T]$ , one has  $(x_n)_{n \in \mathbb{N}}$  has a convergent subsequence.  $\square$

**Theorem 3.1** Assume  $(A_1)$ ,  $(A_2)$ ,  $(B_1)$ ,  $(B_2)$ ,  $(B_3)$  and the following condition

$(B_5)$  there exist positive constants  $l_1, l_2, R, \sigma \leq p$ , such that for  $\|x\| = R$ ,

$$-F(t, x) \leq l_1|x|^\sigma + l_2,$$

and

$$T^{1-\frac{\sigma}{p}}l_1R^\sigma + l_2T < \frac{\min\{\|a\|_m, \frac{1}{M}\}}{p}R^p,$$

hold, then, BVP(1.1) has at least two periodic solutions  $x_0, x_1$  and  $\|x_0\| < R$ .

**Proof.** First, we show for the constant  $R > 0$  given in  $(B_4)$ ,  $\varphi$  has a local minimum point  $x_0$  in  $B_R = \{x \in W^{1,p}[0, T]: \|x\| < R\}$ . There are two steps.

Step 1, we show  $x_0 \in \bar{B}_R$ . Obviously,  $\bar{B}_R$  is bounded and weakly sequentially closed. In the following, we show  $\varphi$  is weakly sequentially lower semi-continuous on  $\bar{B}_R$ . Let

$$\varphi_1(x) = \int_0^T \left[ \frac{1}{p}a(t)|x(t)|^p + \int_0^{x'} G(u)du \right] dt, \quad \varphi_2(x) = \int_0^T F(t, x)dt + \sum_{i=1}^k \int_0^{x(t_i)} I_i(t)dt.$$

Assume  $x_n \rightarrow x$  in  $W^{1,p}[0, T]$ , then,  $\|x_n' - x'\|_{L^p} \rightarrow 0, \|x_n - x\|_{L^p} \rightarrow 0$ . Hence,

$$\begin{aligned} |\varphi_1(x_n) - \varphi_1(x)| &\leq \frac{1}{p} \|a\|_\infty \int_0^T ||x_n|^p - |x|^p| dt + \int_0^T \left| \int_{x'}^{x_n'} G(u)du \right| dt \\ &\leq \frac{1}{p} \|a\|_\infty \int_0^T ||x_n|^p - |x|^p| dt + \frac{1}{mp} \int_0^T ||x_n'|^p - |x'|^p| dt. \end{aligned}$$

Define  $h(x) = |x|^p, p > 1$ . It is obvious that  $h \in C^1[0, T]$ . By applying the Mid-value Theorem, there exists  $\zeta(t)$  satisfying  $0 \leq \zeta(t) \leq 1$  such that

$$||x_n'|^p - |x'|^p| = p|x'(t) + \zeta(t)(x_n'(t) - x'(t))|^{p-1} \times |x_n'(t) - x'(t)|.$$

Moreover, there exists  $\delta > 0$ , such that

$$|x'(t) + \zeta(t)(x_n'(t) - x'(t))|^{p-1} \leq \delta[|x'(t)|^{p-1} + |x_n'(t) - x'(t)|^{p-1}] \in L^q([0, T]).$$

Hence,

$$\begin{aligned} \int_0^T \left| \int_{x'}^{x_n'} G(u)du \right| dt &\leq \left[ \frac{\delta}{mp} \int_0^T |x'(t)|^{p-1} |x_n'(t) - x'(t)| dt + \int_0^T |x_n'(t) - x'(t)|^p dt \right] \\ &\leq \frac{\delta}{mp} \left[ \left( \int_0^T |x'(t)|^p dt \right)^{\frac{1}{q}} \left( \int_0^T |x_n'(t) - x'(t)|^p dt \right)^{\frac{1}{p}} + \int_0^T |x_n'(t) - x'(t)|^p dt \right] \end{aligned}$$

$\rightarrow 0, n \rightarrow \infty$

Here,  $\frac{1}{q} + \frac{1}{p} = 1$ . With the same discussion above, one has  $\int_0^T ||x_n|^p - |x|^p| dt \rightarrow 0$  as  $n \rightarrow \infty$ . Hence,  $\varphi_1$  is continuous. Since  $\varphi_1$  is convex, then,  $\varphi_1$  is weakly lower semi-continuous. Assume  $x_n \rightarrow x$  in  $W^{1,p}[0, T]$ , then,  $x_n \rightarrow x$  in  $C[0, T]$ . Hence,  $\varphi_2$  is weakly semi-continuous. Therefore,  $\varphi$  is weakly lower semi-continuous. From Theorem 2.1,  $\varphi$  has a local minimum  $x_0 \in \bar{B}_R$ .

Step 2. If  $x_0 \in \partial B_R$ ,

$$\begin{aligned} \varphi(x_0) &\geq \int_0^T F(t, x_0)dt + \frac{\|a\|_m}{p} \int_0^T |x_0(t)|^p dt + \frac{1}{pM} \int_0^T |x_0'|^p dt \\ &\geq \int_0^T F(t, x_0)dt + \frac{\min\{\|a\|_m, \frac{1}{M}\}}{p} \|x_0\|^p. \end{aligned} \tag{3.10}$$

Moreover,  $\varphi(x_0) < \varphi(0) = \int_0^T F(t, 0)dt = 0$ , that is,

$$\int_0^T [-F(t, x_0)]dt \geq \frac{\min\{\|a\|_m, \frac{1}{M}\}}{p} R^p. \tag{3.11}$$

From  $(B_4)$ ,

$$\int_0^T [-F(t, x_0)] dt \leq l_1 \int_0^T |x_0|^\sigma dt + l_2 T \leq T^{1-\frac{\sigma}{p}} l_1 \|x_0\|^\sigma + l_2 T = T^{1-\frac{\sigma}{p}} l_1 R^\sigma + l_2 T,$$

together with (3.11), we reach a contradiction. Hence,  $x_0 \in B_R$ , that is  $x_0$  is a critical point of  $\varphi$ .

In the following, we show there exists  $x_1$  satisfying  $\|x_1\| > R$  and  $\varphi(x_1) < \min_{x \in \partial B_R} \varphi(x)$ . From  $(B_2)$ , for  $\forall M > 0$ , there exists  $C_M > 0$  such that

$$F(t, x) \leq -M|x|^p, \quad |x| > C_M.$$

For large  $\lambda > 0$ ,  $M > 0$ , one has

$$\begin{aligned} \varphi(\lambda) &= \int_0^T F(t, \lambda) dt + \lambda^p \int_0^T \frac{1}{p} a(t) dt + \sum_{i=1}^k \int_0^\lambda I_i(t) dt \\ &\leq \lambda^p \int_0^T \frac{1}{p} a(t) dt - M\lambda^p T + \sum_{i=1}^k (b_i \lambda + c_i \lambda^2) \rightarrow -\infty. \end{aligned}$$

Hence, by the mountain pass theorem, one obtain another periodic solution  $x_1$ .

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