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$bu_*(B\mathbb{Z}/p)^n$ AS A GRADED GROUP

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Abstract. Let p be a prime. We calculate $bu_*(B\mathbb{Z}/p)^n$, the connective unitary $K-$theory of the $n-$fold smash product of the classifying space for the cyclic group of order p, as a graded group using a Künneth formula short exact sequence for $n = 2$ and inductively for any $n \geq 2$. While this smashing is in progress some other spectra appear, for instance, the spectrum $H\mathbb{Z}/p \wedge (B\mathbb{Z}/p)^r$ for $r < n$. In order to producing a new homotopy equivalent to $bu \wedge (B\mathbb{Z}/p)^n$, we need to find a homotopy equivalence which simplifies the spectrum $H\mathbb{Z}/p \wedge (B\mathbb{Z}/p)^r$.

Keywords: The connective unitary K-theory;a Künneth formula short exact sequence.

1. Introduction

Let bu_* denote connective unitary K-homology on the stable homotopy category of CW spectra [1] so that if X is a space without a basepoint its unreduced bu-homology is $bu_*(\Sigma^\infty X_+)$, the homology of the suspension spectrum of the disjoint union of X with a base-point. In particular $bu_*(\Sigma^\infty S^0) = \mathbb{Z}[u]$ where $\deg(u) = 2$.

For a prime number p, in [4], we calculate the connective unitary K-theory of the smash product of two copies of the classifying space for the cyclic group of order p, $bu_*(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)$, using a Künneth formula short exact sequence, which is isomorphic to $bu_*(\vee_{i=1}^{p-1}\Sigma^{2i}B\mathbb{Z}/p)$ in odd degrees and isomorphic to $\pi_*(\Sigma^2 H\mathbb{Z}/p[u,v,w]/w^{p-1})$ in even degrees.

In this section we shall merely introduce the Künneth formula sequence which implies our results, in case $n = 2$, upon taking homotopy groups.

In §2 we construct the maps which induce the isomorphisms in 1.3. Inductively, by replacing $B\mathbb{Z}/p \wedge B\mathbb{Z}/p$ by $(B\mathbb{Z}/p)^{\wedge n}$, the $n-$fold smash product of $B\mathbb{Z}/p$, for $n > 2$, we derive a homotopy equivalence of spectra involving $bu \wedge (B\mathbb{Z}/p)^{\wedge n}$, which in the left-hand side and other spectra appearing while this smashing is in progress, for instance, the spectrum $H\mathbb{Z}/p \wedge (B\mathbb{Z}/p)^{\wedge r}$ for $r < n$. In order to produce this homotopy equivalence, we need to find first a homotopy equivalence which simplifies the spectrum $H\mathbb{Z}/p \wedge (B\mathbb{Z}/p)^{\wedge r}$, and start with $r = 1$.

Following this homotopy and by a helpful corollary 3.6 in §3 we apply $\pi_*(-)$, the homotopy group, for this homotopy equivalence to compute $bu_*(B\mathbb{Z}/p)^{\wedge n}$ as a graded group for any prime p.

Let us fix some notations that we will use for this paper.

Notation 1.1.

- For $n \geq 1$, we write P_n for $(B\mathbb{Z}/p)^{\wedge n}$, the $n-$fold smash product of $B\mathbb{Z}/p$. In particular, $P_1 = B\mathbb{Z}/p$.
- For $i, j > 0$, we write $\Lambda_{i,j}$ for $2i + 2j - 2$.

Theorem 1.2. [4] For any CW-complex X, we have a natural exact sequence [which is called the Künneth formula sequence]

$$0 \to bu_*(P_1) \otimes_{\mathbb{Z}_p[u]} bu_*(X) \to bu_*(P_1 \wedge X) \to \mathrm{Tor}^1_{\mathbb{Z}_p[u]}(bu_*(P_1), bu_*(X))[-1] \to 0.$$

Corollary 1.3. [4]

$$bu_{2*+1}(P_2) \cong bu_{2*+1}(\vee_{i=1}^{p-1}\Sigma^{2i}P_1) \text{ and}$$
$$bu_{2*}(P_2) \cong \pi_{2*}(\vee_{i=0}^{p-2} \vee_{j,k>0} \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p).$$

In order to compute $bu_*(P_n)$ as a graded group we need to construct the maps which induce the above isomorphisms and inductively, extend these results to construct a spectrum which is homotopy equivalent to $bu \wedge P_n$ for each $n \geq 2$.

2. The homotopy equivalence

2.1. As we have just seen, by the Künneth sequence we can compute $bu_*(P_2)$, which is isomorphic to $bu_*(\vee_{i=1}^{p-1}\Sigma^{2i}P_1)$ in odd degrees and isomorphic to $\pi_*(\Sigma^2 H\mathbb{Z}/p[u,v,w]/w^{p-1})$ in even degrees. In this section we will construct the maps which induce these isomorphisms. we will extend these results to construct a spectrum which is homotopy equivalent to $bu \wedge P_n$ for each $n \geq 2$.

Remark 2.2.

- $\vee_{i\geq 0}\Sigma^{ki}H\mathbb{Z}/p \simeq H\mathbb{Z}/p[v]$, where $\deg(v) = k$ and $\vee_{i=0}^{p-2}\Sigma^{2i}H\mathbb{Z}/p \simeq H\mathbb{Z}/p[v]/(v^{p-1})$ where $\deg(v) = 2$.
- For $\deg(v) = 2(p-1)$, $\vee_{i=0}^{p-2}\Sigma^{2i}H\mathbb{Z}/p[v] \simeq H\mathbb{Z}/p[u]$ where $\deg(u) = 2$. So

$$\vee_{k=0}^{p-2} \vee_{i,j>0} \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p \simeq \vee_{k=0}^{p-2} \vee_{i,j\geq 0} \Sigma^{2i+2j+2k+2} H\mathbb{Z}/p \simeq \Sigma^2 H\mathbb{Z}/p[u_1,u_2,u_3]/(u_3^{p-1}),$$

where $\deg(u_1) = \deg(u_2) = \deg(u_3) = 2$. We are not claiming any ring structure related to $\mathbb{Z}/p[u_1,u_2,u_3]/(u_3^{p-1})$, it is just a nice way to keep track of all the copies.

Lemma 2.3.
For a prime number p, there is a map $bu \wedge P_2 \to \vee_{i=1}^{p-1}\Sigma^{2i}bu \wedge P_1$, which induces an isomorphism on homotopy groups in odd degrees.

Proof By [3] , we have a homotopy equivalence $bu \wedge BS^1 \simeq \bigvee_{n>0} bu \wedge S^{2n}$. Denote by $\acute{\rho}$: $bu \wedge BS^1 \to \bigvee_{i=1}^{p-1} S^{2i} \wedge bu$ the projection. Therefore we can construct the required map by the composition of the following maps

$$bu \wedge P_2 \xrightarrow{bu\wedge i\wedge P_1} bu \wedge BS^1 \wedge P_1 \xrightarrow{\acute{\rho}\wedge P_1} \vee_{i=1}^{p-1}\Sigma^{2i}bu \wedge P_1,$$

where $\alpha : B\mathbb{Z}/p \to BS^1$ is the classifying space of the inclusion map $\mathbb{Z}/p \to S^1$.

Let $f_n : S^{2n+1} \to bu \wedge P_2$ represents the generator of $bu_{2n+1}(P_2)$, so we have a composition $F_n : S^{2n+1} \to bu \wedge P_2 \to \vee_{i=1}^{p-1}\Sigma^{2i}bu \wedge P_1$ which represents the generator of $bu_{2n+1}(\vee_{i=1}^{p-1}\Sigma^{2i}bu \wedge P_1)$, and this shows that the map $bu \wedge P_2 \to \vee_{i=1}^{p-1}\Sigma^{2i}bu \wedge P_1$ induces an isomorphism in homotopy groups in odd degrees as in 1.3.

□

For an abelian group G, we have $(HG)_n = K(G,n)$ for $n \geq 0$ where $K(G,n)$ is the *Eilenberg-MacLane space*. By [2, p.393], for all CW-complexes X and all $n > 0$ there is a natural bijection $T : [X, K(G,n)] \cong H^n(X;G)$, where T has the form $T([f]) = f^*(x)$ for $x \in H^n(K(G,n);G)$.

Lemma 2.4.

For a prime number p, there is a map $bu \wedge P_2 \to \vee_{k=0}^{p-2} \vee_{i,j>0} \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p$, which induces an isomorphism on homotopy groups in even degrees.

Proof

Let $t = \alpha \otimes \beta \in H^{\Lambda_{i,j}+2k}(P_2;\mathbb{Z}/p)$ and $g_t : P_2 \to \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p$, where α, β, respectively, represent generators of mod p cohomology of P_1 of dimension m and n such that $m+n = \Lambda_{i,j}+2k$, and g_t represents t, where

$$H^{\Lambda_{i,j}+2k}(P_2;\mathbb{Z}/p) \cong [P_2, \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p]$$

Let ψ be the composition of

$$bu \wedge H\mathbb{Z}/p \xrightarrow{\rho\wedge 1} H\mathbb{Z}/p \wedge H\mathbb{Z}/p \xrightarrow{\mu} H\mathbb{Z}/p$$

where $\rho : bu \to H\mathbb{Z}/p$ represents a generator of $(H\mathbb{Z}/p)^0(bu) \cong \mathbb{Z}/p$. The required map is the composition of the following maps,

$$bu \wedge P_2 \xrightarrow{bu\wedge(\vee_{k=0}^{p-2}\vee_{i,j>0} g_t)} bu \wedge (\vee_{k=0}^{p-2} \vee_{i,j>0} \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p) \simeq$$
$$\vee_{k=0}^{p-2} \vee_{i,j>0} \Sigma^{\Lambda_{i,j}+2k} bu \wedge H\mathbb{Z}/p \xrightarrow{\vee_{k=0}^{p-2}\vee_{i,j>0}\Sigma^{\Lambda_{i,j}+2k}\psi} \vee_{k=0}^{p-2} \vee_{i,j>0} \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p,$$

and using 1.3, we see that this map induces an isomorphism on homotopy groups in even degrees. □

By 2.3 and 2.4 we have the following result.

Theorem 2.5. For any prime number p, there is a homotopy equivalence

$$bu \wedge P_2 \simeq \vee_{i=1}^{p-1} \Sigma^{2i} bu \wedge P_1 \vee (\vee_{k=0}^{p-2} \vee_{i,j>0} \Sigma^{\Lambda_{i,j}+2k} H\mathbb{Z}/p).$$

Remark 2.6. From 2.2, the homotopy equivalence in 2.5 can be written as

$$bu \wedge P_2 \simeq \vee_{i=1}^{p-1} \Sigma^{2i} bu \wedge P_1 \vee \Sigma^2 H\mathbb{Z}/p[u,v,w]/(w^{p-1})$$

where $\deg(u) = \deg(v) = \deg(w) = 2$.

Inductively, by replacing P_2 by P_n, for $n > 2$, we get another homotopy equivalence, where the left-hand side is $bu \wedge P_n$. While this smashing is in progress some other spectra appear, for instance, the spectrum $H\mathbb{Z}/p \wedge P_r$ for $r < n$. In order to conclude this section by producing a new homotopy equivalence, we need to find first a homotopy equivalence which simplifies the spectrum $H\mathbb{Z}/p \wedge P_r$. Let us start with $r = 1$.

Lemma 2.7. For any prime p, $H\mathbb{Z}/p \wedge P_1 \simeq \bigvee_{i>0} \Sigma^i H\mathbb{Z}/p$.

Proof Let $f_i : S^i \to H\mathbb{Z}/p \wedge P_1$ represents a generator of $\pi_i(H\mathbb{Z}/p \wedge P_1) \cong H_i(P_1, \mathbb{Z}/p) \cong [S^i, H\mathbb{Z}/p \wedge P_1]$, and let $f : \bigvee_{i>0} S^i \to H\mathbb{Z}/p \wedge P_1$ such that $f|_{S^i} = f_i$. We construct F as the composition of these maps

$$\bigvee_{i>0} H\mathbb{Z}/p \wedge S^i \simeq \bigvee_{i>0} \Sigma^i H\mathbb{Z}/p \xrightarrow{H\mathbb{Z}/p\wedge f} H\mathbb{Z}/p \wedge H\mathbb{Z}/p \wedge P_1 \xrightarrow{\mu\wedge P_1} H\mathbb{Z}/p \wedge P_1.$$

Let g represents a generator of $\pi_j(\bigvee_{i>0} H\mathbb{Z}/p \wedge S^i)$, so $g : S^j \to \bigvee_{i>0} H\mathbb{Z}/p \wedge S^i$ and the composition

$$S^j \to \bigvee_{i>0} H\mathbb{Z}/p \wedge S^i \to H\mathbb{Z}/p \wedge P_1$$

represents a generator of $\pi_j(H\mathbb{Z}/p\wedge P_1) = [S^j, H\mathbb{Z}/p\wedge P_1]$. Thus the map F induces an isomorphism $F_* : \pi_{2k+1}(\bigvee_{i>0} H\mathbb{Z}/p \wedge S^i) \cong \mathbb{Z}/p \cong \pi_{2k+1}(H\mathbb{Z}/p \wedge P_1)$ for all k, and Whitehead's Theorem shows the required result. □

Remark 2.8. For $r = 1$, smashing P_r with the Eilenberg-MacLane spectrum of the group $\mathbb{Z}/p$ gives a wedge of suspensions of this spectrum of the same group. This can be written also as $H\mathbb{Z}/p \wedge P_1 \simeq \widetilde{H}V_1$, where $\widetilde{H}$ refers to the reduced case, $V_1 = \mathbb{Z}/p[\alpha_1]$ and $\deg(\alpha_1) = 1$. Next we will explain the corresponding result for $r \geq 1$.

Corollary 2.9. For $r \geq 1$, $H\mathbb{Z}/p \wedge P_r \simeq \vee_{n_1,n_2,\ldots,n_r>0}\Sigma^{n_1+n_2+\cdots+n_r}H\mathbb{Z}/p$.

Proof It follows by induction from the previous result that

$$\begin{aligned} H\mathbb{Z}/p \wedge P_r &= H\mathbb{Z}/p \wedge P_{r-1} \wedge P_1 \\ &\cong \vee_{n_1,n_2,\ldots,n_{r-1}>0}\Sigma^{n_1+n_2+\cdots+n_{r-1}}(H\mathbb{Z}/p \wedge P_1) \\ &\cong \vee_{n_1,n_2,\ldots,n_r>0}\Sigma^{n_1+n_2+\cdots+n_r}H\mathbb{Z}/p. \end{aligned}$$

□

Theorem 2.10.
For $n \geq 2$ and a prime number p,

$$bu \wedge P_n \simeq \Big(\bigvee_{i_1,i_2,\ldots,i_{n-1}=1}^{p-1} \Sigma^{2i_1+2i_2+\cdots+2i_{n-1}} bu \wedge P_1 \Big) \vee \bigvee_{i=1}^{n-1} \Sigma^{n+i-1} HV_{n,i}$$

where $V_{n,i} = \mathbb{Z}/p[u_1, u_2, \ldots, u_{i+2}, v_1, v_2, \ldots, v_{n-i-1}]/(u_1^{p-1}, u_2^{p-1}, \ldots, u_i^{p-1})$, $\deg(u_j) = 2$ and $\deg(v_j)$ 1 for all j.

Proof The proof is by induction on n, where the case $n = 2$ is considered in 2.5 and 2.6 and the result agrees with the above statement.

Now let us assume that the statement is true for $n-1$. By 2.6 we have

$$\begin{aligned} bu \wedge P_n = bu \wedge P_2 \wedge P_{n-2} &\simeq \Big(\vee_{j=1}^{p-1} \Sigma^{2j} bu \wedge P_{n-1}\Big) \vee \Big(\Sigma^2 H\mathbb{Z}/p[u_1,u_2,u_3]/(u_1^{p-1}) \wedge P_{n-2}\Big) \\ &\simeq \vee_{j=1}^{p-1}\Sigma^{2j}\Big(\vee_{i_1,\ldots,i_{n-2}=1}^{p-1} \Sigma^{2i_1+\cdots+2i_{n-2}} bu \wedge P_1 \vee \vee_{i=1}^{n-2}\Sigma^{n+i-2}HV_{n-1,i}\Big) \vee \Big(\Sigma^2 HV_{2,1} \wedge P_{n-2}\Big) \end{aligned}$$

By 2.2 we have

$$\begin{aligned} \vee_{i=1}^{n-2} \vee_{j=1}^{p-1} \Sigma^{n+i-2+2j} HV_{n-1,i} &= \vee_{i=1}^{n-2} \vee_{j=0}^{p-2} \Sigma^{n+i+2j} HV_{n-1,i} \\ &\simeq \vee_{i=1}^{n-2}\Sigma^{n+i}H\mathbb{Z}/p[u_1,\ldots,u_{i+3},v_1,\ldots,v_{n-i-2}]/(u_1^{p-1},\ldots,u_{i+1}^{p-1}) \end{aligned}$$

where the right side spectrum is equal to $\vee_{i=2}^{n-1}\Sigma^{n+i-1}HV_{n,i}$. And by 2.9, for $r = n-2$, we have

$$\Sigma^2 HV_{2,1} \wedge P_{n-2} \simeq \Sigma^n H\mathbb{Z}/p[u_1,u_2,u_3,v_1,v_2,\ldots,v_{n-2}]/(u_1^{p-1})$$

where the right side spectrum is equal to $\Sigma^{n+i-1}HV_{n,i}$ when $i=1$. Therefore

$$bu \wedge P_n \simeq \Big(\bigvee_{i_1,i_2,\ldots,i_{n-1}=1}^{p-1} \Sigma^{2i_1+2i_2+\cdots+2i_{n-1}} bu \wedge P_1 \Big) \vee \bigvee_{i=2}^{n-1} \Sigma^{n+i-1}HV_{n,i} \vee \bigvee_{i=1} \Sigma^{n+i-1}HV_{n,i}$$
$$= \Big(\bigvee_{i_1,i_2,\ldots,i_{n-1}=1}^{p-1} \Sigma^{2i_1+2i_2+\cdots+2i_{n-1}} bu \wedge P_1 \Big) \vee \bigvee_{i=1}^{n-1} \Sigma^{n+i-1}HV_{n,i}.$$

□

3. $bu_*(P_n)$ as a Graded Group

3.1. In this section we are going to compute $bu_*(P_n)$ as a graded group, first for $p=2$ and after that for any prime p. This calculations follow the homotopy equivalence in 2.10.

Notation 3.2. For $n \geq 2$ and $p=2$, the homotopy equivalence which is in 2.10 can be written as

$$bu \wedge P_n \simeq \Sigma^{2(n-1)} bu \wedge P_1 \vee \vee_{i=1}^{n-1} \vee_{j_1,j_2,k_1,k_2,\ldots,k_{i-1}>0} \Sigma^{2(n-i-1)+\Lambda+\Sigma_{j=1}^{i-1}k_j} H\mathbb{Z}/2$$

where $\Lambda = 2j_1 + 2j_2 - 2$, see 1.1.

To compute $bu_*(P_n)$ as a graded group, we apply $\pi_*(-)$ for the previous homotopy. In [4] the graded group $\pi_*(\Sigma^{2(n-1)} bu \wedge P_1)$ is computed, which is non-zero just in odd degrees. So to complete this calculation we need to compute the homotopy group of the rest of the wedges of the above homotopy, that is, $\pi_*(\vee_{i=1}^{n-1} \vee_{j_1,j_2,k_1,k_2,\ldots,k_{i-1}>0} \Sigma^{2(n-i-1)+\Lambda+\Sigma_{j=1}^{i-1}k_j} H\mathbb{Z}/2)$.

Now we are going to introduce and prove a supporting corollary for this calculation.

Corollary 3.3. Let $X_*^{m,k_1,k_2,\ldots,k_n} = \oplus_{i,j>0}\pi_*(\Sigma^{2m+\Lambda_{i,j}+\Sigma_{a=1}^{n}k_a} H\mathbb{Z}/2)$, where $n, m,$ and $k_a \geq 0$ and $\Lambda_{i.j}$ as in 1.1. Then, for $t, \ell \geq 0$

$$\oplus_{k_1,k_2,\ldots,k_{2\ell}>0 \text{ odd}} \oplus_{k_{2\ell+1},\ldots,k_n>0 \text{ even}} X_{2t}^{m,k_1,k_2,\ldots,k_n} = (\mathbb{Z}/2)^{\binom{t-m+\ell}{n+1}},$$

and

$$\oplus_{k_1,k_2,\ldots,k_{2\ell+1}>0 \text{ odd}} \oplus_{k_{2\ell+2},\ldots,k_n>0 \text{ even}} X_{2t+1}^{m,k_1,k_2,\ldots,k_n} = (\mathbb{Z}/2)^{\binom{t-m+\ell+1}{n+1}}.$$

Proof Inductively, we will prove the first claim, in degree $2t$, and the analogous calculations for the other claims are similar.

Firstly,

$$X_{2t}^{m,k_1,k_2,\ldots,k_n} = (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=1}^{n}\frac{k_a}{2}}{1}}$$

and analogously,

$$X_{2t+1}^{m,k_1,k_2,\ldots,k_n} = (\mathbb{Z}/2)^{\binom{t-m+\frac{1}{2}-\Sigma_{a=1}^{n}\frac{k_a}{2}}{1}}.$$

So

$$\oplus_{k_1>0 \text{ even}} X_{2t}^{m,k_1,k_2,\ldots,k_n} = (\mathbb{Z}/2)^{\Sigma_{s=0}^{t-m-1-\Sigma_{a=2}^{n}\frac{k_a}{2}}\binom{s}{1}} = (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=2}^{n}\frac{k_a}{2}}{2}}.$$

Then, inductively, if we assume that

$$\oplus_{k_1,k_2,\ldots,k_\beta>0 \text{ even}} X_{2t}^{m,k_1,k_2,\ldots,k_n} = (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=\beta+1}^{n}\frac{k_a}{2}}{\beta+1}},$$

then we have

$$\oplus_{k_1,k_2,\dots,k_{\beta+1}>0 \text{ even}} X_{2t}^{m,k_1,k_2,\dots,k_n} = \oplus_{k_{\beta+1}>0 \text{ even}} (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=\beta+1}^{n}\frac{k_a}{2}}{\beta+1}}$$
$$= (\mathbb{Z}/2)^{\Sigma_{s=0}^{t-m-1-\Sigma_{a=\beta+2}^{n}\frac{k_a}{2}}\binom{s}{\beta+1}} = (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=\beta+2}^{n}\frac{k_a}{2}}{\beta+2}}.$$

Similarly, we can calculate that,

$$\oplus_{k_1,k_2>0 \text{ odd}} \oplus_{k_3,k_4\dots,k_n>0 \text{ even}} X_{2t}^{m,k_1,k_2,\dots,k_n} = \oplus_{k_1,k_2>0 \text{ odd}} (\mathbb{Z}/2)^{\binom{t-m-\frac{k_1+k_2}{2}}{n-1}}$$
$$= \oplus_{k_1>0 \text{ odd}} (\mathbb{Z}/2)^{\Sigma_{s=0}^{t-m-\frac{1+k_1}{2}}\binom{s}{n-1}}$$
$$= \oplus_{k_1>0 \text{ odd}} (\mathbb{Z}/2)^{\binom{t-m-\frac{k_1}{2}+\frac{1}{2}}{n}}$$
$$= (\mathbb{Z}/2)^{\Sigma_{s=0}^{t-m}\binom{s}{n}} = (\mathbb{Z}/2)^{\binom{t-m+1}{n+1}}.$$

Hence

$$\oplus_{k_1,k_2,\dots,k_{2\ell}>0 \text{ odd}} \oplus_{k_{2\ell+1},\dots,k_n>0 \text{ even}} X_{2t}^{m,k_1,k_2,\dots,k_n} = \oplus_{k_1,k_2,\dots,k_{2\ell}>0 \text{ odd}} (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=1}^{2\ell}\frac{k_a}{2}}{n-2\ell+1}}$$
$$= \oplus_{k_1,k_2,\dots,k_{2\ell-1}>0 \text{ odd}} (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=1}^{2\ell-1}\frac{k_a}{2}+\frac{1}{2}}{n-2\ell+2}}$$
$$= \oplus_{k_1,k_2,\dots,k_{2\ell-2}>0 \text{ odd}} (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=1}^{2\ell-2}\frac{k_a}{2}+1}{n-2\ell+3}}$$
$$= \vdots$$
$$= \oplus_{k_1,k_2,\dots,k_{2\ell-2\eta}>0 \text{ odd}} (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=1}^{2\ell-2\eta}\frac{k_a}{2}+\eta}{n-2\ell+2\eta+1}}.$$

When $\eta = \ell - 1$, the right side is equal to

$$\oplus_{k_1,k_2>0 \text{ odd}} (\mathbb{Z}/2)^{\binom{t-m-\Sigma_{a=1}^{2}\frac{k_a}{2}+\ell-1}{n-1}} = (\mathbb{Z}/2)^{\binom{t-m+\ell}{n+1}}.$$

This completes the proof. □

Lemma 3.4. Let $n_1 \geq 0$. Then

$$bu_s(P_n) \cong$$
$$\begin{cases} (\mathbb{Z}/2)^{\sum_{j=0}^{n-2}\sum_{i=0}^{n_1-1}\binom{j}{2i}\binom{t-n+j+i+2}{j+1}}, & \text{when } s=2t,\ n=2n_1 \text{ or } n=2n_1+1, \\ \mathbb{Z}/2^{t-n+2} \oplus (\mathbb{Z}/2)^{\sum_{j=0}^{n-2}\sum_{i=0}^{n_1-2}\binom{j}{2i+1}\binom{t-n+j+i+3}{j+1}}, & \text{when } s=2t+1,\ n=2n_1, \\ \mathbb{Z}/2^{t-n+2} \oplus (\mathbb{Z}/2)^{\sum_{j=0}^{n-2}\sum_{i=0}^{n_1-1}\binom{j}{2i+1}\binom{t-n+j+i+3}{j+1}}, & \text{when } s=2t+1,\ n=2n_1+1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof If we apply homotopy groups to the homotopy equivalence in 3.2, we get

$$bu_*(P_n) \cong bu_*(\Sigma^{2(n-1)}P_1) \oplus \oplus_{i=1}^{n-1} \oplus_{k_1,k_2,\dots,k_{i-1}>0} X_*^{n-i-1,k_1,k_2,\dots,k_{i-1}}.$$

The factor $bu_*(\Sigma^{2(n-1)}P_1)$ is concentrated just in odd degrees, see [4], whereas the factor

$$\oplus_{k_1,k_2,\ldots,k_{i-1}>0}X_*^{n-i-1,k_1,k_2,\ldots,k_{i-1}}$$

is concentrated just in even degrees when $i=1$ and in both odd and even degrees when $i>1$. By 3.3, the result follows. $\square$

3.5. When p is odd we write bu for the connective unitary K-theory with p-adic integers coefficients where $bu \simeq \vee_{i=0}^{p-2}\Sigma^{2i}lu$, and lu is called the Adams summand, $lu_*(\Sigma^\infty S^0)\cong \mathbb{Z}_p[v]$ and $\deg(v)=2p-2$. By [4] we have $lu_{2k(p-1)+2i-1}(\Sigma^\infty B\mathbb{Z}/p)\cong \mathbb{Z}/p^{k+1}$ for $i=1,\ldots,p-1$.

For $n\geq 2$ and any prime p, the homotopy equivalence which is in 2.10 can be written as

$$bu\wedge P_n \simeq \Big(\bigvee_{i_1,i_2,\ldots,i_{n-1}=1}^{p-1}\Sigma^{2i_1+2i_2+\cdots+2i_{n-1}}bu\wedge P_1\Big)$$
$$\vee \vee_{i=1}^{n-1}\vee_{j_1,j_2,k_1,k_2,\ldots,k_{i-1}>0}\vee_{\lambda_1,\lambda_2,\ldots,\lambda_{n-i}=0}^{p-2}\Sigma^{2(n-i-1)+\Lambda_{j_1,j_2}+\Sigma_{a=1}^{i-1}k_a+2\Sigma_{a=1}^{n-i}\lambda_a}H\mathbb{Z}/p.$$

where $\Lambda_{j_1,j_2}=2j_1+2j_2-2$, see 1.1.

And again to compute $bu_*(P_n)$ as a graded group for any prime p, we apply $\pi_*(-)$ for the previous homotopy. The graded group $\oplus_{i_1,i_2,\ldots,i_{n-1}=1}^{p-1}\pi_{2t+1}(\Sigma^{2i_1+2i_2+\cdots+2i_{n-1}}bu\wedge P_1)\cong\oplus_{i_1,i_2,\ldots,i_{n-1}=1}^{p-1}\oplus_{j=0}^{p-2}\mathbb{Z}/p^{k+1}$ where $t=k(p-1)+\sum_{\beta=1}^{n-1}i_\beta+j+i$ and $i=0,1,2,\ldots,p-2$, which is non-zero just in odd degrees. So to complete this calculation we need to compute the homotopy group of the rest of the wedges of the above homotopy, that is,

$$\pi_*(\vee_{i=1}^{n-1}\vee_{j_1,j_2,k_1,k_2,\ldots,k_{i-1}>0}\vee_{\lambda_1,\lambda_2,\ldots,\lambda_{n-i}=0}^{p-2}\Sigma^{2(n-i-1)+\Lambda_{j_1,j_2}+\Sigma_{a=1}^{i-1}k_a+2\Sigma_{a=1}^{n-i}\lambda_a}H\mathbb{Z}/p).$$

Similarly, we need to introduce a supporting corollary for this calculation.

Corollary 3.6. Let $X_*^{m,k_1,k_2,\ldots,k_n,\lambda}=\oplus_{i,j>0}\pi_*(\Sigma^{2m+\Lambda_{i,j}+\Sigma_{a=1}^{n}k_a+2\Sigma_{a=1}^{\alpha}\lambda_a}H\mathbb{Z}/p)$, where

$n,m,\alpha,k_a$and $\lambda_a\geq 0$ and $\Lambda_{i,j}$ as in 1.1. Then, for $t,\ell\geq 0$

$$\oplus_{k_1,k_2,\ldots,k_{2\ell}>0 \text{ odd}}\oplus_{k_{2\ell+1},\ldots,k_n>0 \text{ even}}X_{2t}^{m,k_1,k_2,\ldots,k_n,\lambda}=(\mathbb{Z}/p)^{\binom{t-m+\ell-\Sigma_{a=1}^{\alpha}\lambda_a}{n+1}},$$

and

$$\oplus_{k_1,k_2,\ldots,k_{2\ell+1}>0 \text{ odd}}\oplus_{k_{2\ell+2},\ldots,k_n>0 \text{ even}}X_{2t+1}^{m,k_1,k_2,\ldots,k_n,\lambda}=(\mathbb{Z}/p)^{\binom{t-m+\ell+1-\Sigma_{a=1}^{\alpha}\lambda_a}{n+1}}.$$

The proof is similar to 3.3, where

$$X_{2t}^{m,k_1,k_2,\ldots,k_n,\lambda}=(\mathbb{Z}/p)^{\binom{t-m-\Sigma_{a=1}^{n}\frac{k_a}{2}-\Sigma_{a=1}^{\alpha}\lambda_a}{1}}$$

and

$$X_{2t+1}^{m,k_1,k_2,\ldots,k_n,\lambda}=(\mathbb{Z}/p)^{\binom{t-m+\frac{1}{2}-\Sigma_{a=1}^{n}\frac{k_a}{2}-\Sigma_{a=1}^{\alpha}\lambda_a}{1}}.$$

Remark 3.7. We write $\Gamma(k,t)$ for $\oplus_{i_1,i_2,\ldots,i_{n-1}=1}^{p-1}\oplus_{j=0}^{p-2}\mathbb{Z}/p^{k+1}$ when $t=k(p-1)+\sum_{\beta=1}^{n-1}i_\beta+j+i$ and $i=0,1,2,\ldots,p-2$

By applying the homotopy group to the homotopy equivalence in 3.5, and by 3.6 we get the following result.

Theorem 3.8. *Let $n_1 \geq 0$. Then for a prime p,*

$$bu_s(P_n) \cong$$

$$\begin{cases} (\mathbb{Z}/p)^{\sum_{j=0}^{n-2} \sum_{\lambda_1,\lambda_2,\ldots,\lambda_{n-j-1}=0}^{p-2} \sum_{i=0}^{n_1-1} \binom{j}{2i}\binom{t-n+j+i+2-\Sigma_{a=1}^{n-j-1}\lambda_a}{j+1}}, & \text{when } s=2t,\ n=2n_1 \\ & \text{or } n=2n_1+1, \\ \Gamma(k,t)\oplus(\mathbb{Z}/p)^{\sum_{j=0}^{n-2} \sum_{\lambda_1,\lambda_2,\ldots,\lambda_{n-j-1}=0}^{p-2} \sum_{i=0}^{n_1-2} \binom{j}{2i+1}\binom{t-n+j+i+3-\Sigma_{a=1}^{n-j-1}\lambda_a}{j+1}}, & \text{when } s=2t+1, \\ & n=2n_1, \\ \Gamma(k,t)\oplus(\mathbb{Z}/p)^{\sum_{j=0}^{n-2} \sum_{\lambda_1,\lambda_2,\ldots,\lambda_{n-j-1}=0}^{p-2} \sum_{i=0}^{n_1-1} \binom{j}{2i+1}\binom{t-n+j+i+3-\Sigma_{a=1}^{n-j-1}\lambda_a}{j+1}}, & \text{when } s=2t+1, \\ & n=2n_1+1, \\ 0, & \text{otherwise}. \end{cases}$$

Example 3.9. *Let $p = 2$ and $n = 5$, then by 3.3 we have*

$$bu_{2t}(P_5) = (\mathbb{Z}/2)^{\sum_{j=0}^{3} \sum_{\ell=0}^{1} \binom{j}{2\ell}\binom{t-3+j+\ell}{j+1}}.$$

Similarly, we can calculate that

$$bu_{2t+1}(P_5) \cong \mathbb{Z}/2^{t-3} \oplus (\mathbb{Z}/2)^{\sum_{j=0}^{3} \sum_{\ell=0}^{1} \binom{j}{2\ell+1}\binom{t+j+\ell-2}{j+1}}.$$

For $t = 5$, we have $bu_{10}(P_5) = (\mathbb{Z}/2)^{69}$, the direct sum of 69 copies of $\mathbb{Z}/2$. And $bu_{11}(P_5) = \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{106}$. whereas, in case $p = 3$, $bu_{11}(P_3) = (\mathbb{Z}/3)^4 \oplus (\mathbb{Z}/3^2)^4 \oplus (\mathbb{Z}/3)^{25} = (\mathbb{Z}/3^2)^4 \oplus (\mathbb{Z}/3)^{29}$, and $bu_{10}(P_3) = (\mathbb{Z}/3)^{28}$.

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