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Results in Fuzzy Probabilistic Metric Space

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Abstract

In this paper we prove fixed point theorem in the setting of fuzzy probabilistic metric space using weak commuting mappings and also prove result for rational expression.

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Key words: Fuzzy Probabilistic Metric Space (FPMr Space); weak commuting mappings Common Fixed Points.

Introduction and Preliminaries

Menger [2] in 1942 introduced the notation of the probabilistic metric space. The probabilistic generalization of metric space appears to be well adopted for the investigation of physical quantities and physiological thresholds.

Cho et al. [1] introduced the notation of semi compatible maps in a- topological space. According to them a pair of self-maps (S,T) to be semi compatible if condition (i) $Sy = Ty \Rightarrow STy = TSy$; (ii) the sequence $\{x_n\}$ in X and $x \in X$, $\{Sx_n\} \to x$, $\{Tx_n\} \to x$ then $STx_n = Tx$ as $n \to \infty$, hold. We define semi compatible self-maps in probabilistic metric space by (ii) only. Popa in [3] used the family Φ of implicit function to find the fixed points of two pairs of semi compatible maps in a d complete topological space, where Φ be the family of real continuous function $\phi : (R^+)^4 \to R$ satisfying the properties

(G_h) for every $u \geq 0$, $v \geq 0$ with $\phi(u,v,u,v) \geq 0$ or $\phi(u,v,v,u) \geq 0$ we have $u \geq v$.

(G_u) $\phi(u,u,1,1) \geq 0$ implies that $u \geq 1$

The main object of this paper we prove fixed point theorem in the setting of fuzzy probabilistic metric space using weak compatibility also prove result for rational expression.

Definition 4.2.1 A fuzzy probabilistic metric space (FPM space) is an ordered pair (X, F_α) consisting of a nonempty set X and a mapping F_α from XxX into the collections of all fuzzy distribution functions $F_\alpha \in R$ $\forall \alpha.$ $\in [0,1]$. For x, y $\in$ X we denote the distribution function F_α (x,y) by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F_{\alpha(x,y)}$ at u in R.

The functions $F_{\alpha(x,y)}$ for all $\alpha. \in [0,1]$ assumed to satisfy the following conditions:

(a) $F_{\alpha(x,y)}(u) = 1 \ \forall \ u > 0$ iff $x = y$,

(b) $F_{\alpha(x,y)}(0) = 0 \ \forall$ x , y in X,

(c) $F_{\alpha(x,y)} = F_{\alpha(y,x)} \ \forall$ x , y in X,

(d) If $F_{\alpha(x,y)}(u) = 1$ and $F_{\alpha(y,z)}(v) = 1$ then $F_{\alpha(x,z)}(u+v) = 1$

$\forall$ x , y ,z in X and u, v > 0

Definition 4.2.2 A commutative, associative and non-decreasing mapping t: $[0,1] \times [0,1] \to [0,1]$ is a t-norm if and only if t(a,1)=a for all $a \in [0,1]$, t(0,0)=0 and $t(c,d) \geq t(a,b)$ for $c \geq a$, $d \geq b$.

Definition 4.2.3 A Fuzzy Menger space is a triplet (X,F_α,t), where (X,F_α) is a FPM-space, t is a t-norm and the generalized triangle inequality

$$F_{\alpha(x,z)}(u+v) \geq t\,(F_{\alpha(x,z)}(u), F_{\alpha(y,z)}(v))$$

holds for all x, y, z in X u, v > 0 and α. ∈ [0,1]

The concept of neighborhoods in Fuzzy Menger space is introduced as

Definition 4.2.4 Let (X,F_α,t) be a Fuzzy Menger space. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0,1)$, then (ε,λ) -neighborhood of x, called $U_x\,(\varepsilon,\lambda)$, is defined by

$$U_x\,(\varepsilon,\lambda) = \{y \in X: F_{\alpha(x,y)}(\varepsilon) > (1-\lambda)\}$$

An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda): x \in X, \varepsilon > 0, \alpha. \in [0,1]$and $\lambda \in (0,1)\}$ of neighborhood.

Remark: If t is continuous, then Fuzzy Menger space (X,F_α,t) is a Housdroff space in (ε,λ)-topology.

Let (X,F_α,t) be a complete Fuzzy Menger space and $A \subset X$. Then A is called a bounded set if

$$\lim_{u\to\infty} \inf_{x,y\in A} F_{\alpha(x,y)}\,(u) = 1$$

Definition 4.2.5 A sequence $\{x_n\}$ in (X,F_α,t) is said to be convergent to a point x in X if for every $\varepsilon > 0$and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $x_n \in U_x(\varepsilon,\lambda)$ for all $n \geq N$ or equivalently $F_\alpha\,(x_n, x; \varepsilon) > 1-\lambda$ for all $n \geq N$ and $\alpha \in [0,1]$.

Definition 4.2.6 A sequence $\{x_n\}$ in (X,F_α, t) is said to be cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $F_\alpha(x_n,x_m; \varepsilon) > 1-\lambda$ $\forall$ n, $m \geq N$ for all $\alpha \in [0,1]$.

Definition 4.2.7 A Fuzzy Menger space (X,F_α,t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha \in [0,1]$.

Definition 4.2.8 Let (X,F_α,t) be a Fuzzy Menger space. Two mappings f, g $:X \to X$ are said to be weakly comptable if they commute at coincidence point for all $\alpha \in [0,1]$.

Lemma 4.2.1 Let $\{x_n\}$ be a sequence in a Fuzzy Menger space (X,F_α,t), where t is continuous and $t(p,p) \geq p$ for all $p \in [0,1]$, if there exists a constant k(0,1) such that for all p > 0 and $n \in N$

$$F_\alpha\,(x_n,x_{n+1}; kp) \geq F_\alpha(x_{n-1},x_n; p),$$

for all $\alpha \in [0,1]$ then $\{x_n\}$ is cauchy sequence.

Lemma 4.2.2 If (X,d) is a metric space, then the metric d induces, a mapping F_α: XxX→L defined by F_α (p, q) = H_α(x- d(p, q)), p, q ∈ R for all $\alpha \in [0,1]$. Further if t: $[0,1] \times [0,1] \to [0,1]$ is defined by t(a,b) = min{a,b}, then (X,F_α,t) is a Fuzzy Menger space. It is complete if (X,d) is complete.

Main results

Theorem (1.1): Let (X,F_α,t) be a complete Fuzzy Menger space, where t is continuous and $t(p,p) \geq p$ for all p and α in [0,1].Let S and T be continuous mappings of X , then S and T have a common fixed point in X if there exists continuous mappings A of X into $S(X) \cap T(X)$ which commute weakly with S and T and

$$(1.1)\quad F_\alpha(Ax,Ay,kp) \geq \phi\{F_\alpha(Ty,Ay,p), F_\alpha(Sx,Ax,p), F_\alpha(Sx,Ty,p), F_\alpha(Ax,Ty,p), F_\alpha(Sx,Ay,p)\}$$

for all $x,y \in X$ and p > 0. Then S, T and A have a unique common fixed point.

Proof: We define a sequence $\{x_n\}$ such that

$Ax_{2n} = Sx_{2n-1}$ and $A,x_{2n-1} = Tx_{2n}$ n = 1, 2,--

We shall prove that $\{Ax_n\}$is a Cauchy sequence.

$$F_\alpha\left(Ax_{2n}, Ax_{2n+1}kp\right) \geq \min\left\{\begin{matrix} F_\alpha\left(Tx_{2n+1}, Ax_{2n+1}, p\right), F_\alpha\left(Sx_{2n}, Ax_{2n}, p\right), F_\alpha\left(Sx_{2n}, Tx_{2n+1}, p\right), \\ F_\alpha\left(Ax_{2n}, Tx_{2n+1}, p\right), F_\alpha\left(Sx_{2n}, Ax_{2n+1}, p\right) \end{matrix}\right\}$$

$$F_\alpha\left(Ax_{2n}, Ax_{2n+1}, kp\right) \geq \min\left\{\begin{matrix} F_\alpha\left(Ax_{2n}, Ax_{2n+1}, p\right), F_\alpha\left(Ax_{2n+1}, Ax_{2n}, p\right), F_\alpha\left(Ax_{2n+1}, Ax_{2n}, p\right), \\ F_\alpha\left(Ax_{2n}, Ax_{2n}, p\right), F_\alpha\left(Ax_{2n+1}, Ax_{2n+1}, p\right) \end{matrix}\right\}$$

$$= \min\left\{F_\alpha(Ax_{2n}, Ax_{2n+1}, p), F_\alpha(Ax_{2n+1}, Ax_{2n}, p), F_\alpha(Ax_{2n+1}, Ax_{2n}, p), 1, 1\right\}$$

$$\geq \min\left\{F_\alpha(Ax_{2n-1}, Ax_{2n}, \frac{p}{k}), F_\alpha(Ax_{2n}, Ax_{2n-1}\frac{p}{k}), F_\alpha(Ax_{2n}, Ax_{2n-1}, \frac{p}{k}), 1, 1\right\}$$

Therefore

$$F_\alpha\left(Ax_{2n}, Ax_{2n+1}, kp\right) \geq F_\alpha\left(Ax_{2n-1}, Ax_{2n}, \frac{p}{k}\right)$$

By induction

$$F_\alpha\left(Ax_{2k}, Ax_{2m+1}, kp\right) \geq F_\alpha\left(Ax_{2m}, Ax_{2k-1}, \frac{p}{k}\right)$$

For every k and m in N, Further if 2m +1 > 2k, then

$$F_\alpha\left(Ax_{2k}, Ax_{2m+1}, kp\right) \geq F_\alpha\left(Ax_{2k-1}, Ax_{2m}, \frac{p}{k}\right) \ldots\ldots\ldots\ldots \geq F_\alpha\left(Ax_0, Ax_{2m+1-2k}, \frac{p}{k^{2k}}\right) \text{---- (8.2.1b)}$$

If 2k > 2m+1, then

$$F_\alpha\left(Ax_{2k}, Ax_{2m+1}, kp\right) \geq F_\alpha\left(Ax_{2k-1}, Ax_{2m}, \frac{p}{k}\right) \ldots\ldots\ldots\ldots \geq F_\alpha\left(Ax_{2k-(2m+1)}, Ax_0, \frac{p}{k^{2m+1}}\right) \text{----- (1.1 c)}$$

By simple induction with (1.1 b) and (1.1c) we have

$$F_\alpha\left(Ax_n, Ax_{n+p}, kp\right) \geq F_\alpha\left(Ax_0, Ax_p, \frac{p}{k^n}\right).$$

For n = 2k, p = 2m+1 or n = 2k+1, p = 2m +1

$$F_\alpha\left(Ax_n, Ax_{n+p}, kp\right) \geq F_\alpha\left(Ax_0, Ax_1, \frac{p}{2k^n}\right).F_\alpha\left(Ax_1, Ax_p, \frac{p}{k^n}\right). --- \text{(1.1 d)}$$

If n = 2k, p = 2m or n = 2k+1, p= 2m

For every positive integer p and n in N, by nothing that

$$F_\alpha\left(Ax_0, Ax_p, \frac{p}{k^n}\right) \to 1 \, as \, n \to \infty$$

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete there exists $z \in X$, such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n} = z$$

Science A is weakly commute with S and T ,so Az = Sz = Tz and

Therefore

$$F_\alpha(Ax_n, Az, kp) \geq \inf \left\{ \begin{matrix} F_\alpha(Tz, Az, p), F_\alpha(Sx_n, Ax_n, p), F_\alpha(Sx_n, Tz, p), F_\alpha(Ax_n, Tz, p), \\ F_\alpha(Sx_n, Az, p) \end{matrix} \right\}$$

On taking limit, we get

$$F_\alpha(z, Az, kp) \geq \inf \{F_\alpha(Tz, Az, p), F_\alpha(z, z, p), F_\alpha(z, Tz, p), F_\alpha(z, Tz, p), F_\alpha(z, Az, p)\}$$

$$F_\alpha(z, Az, kp) \geq \inf \{F_\alpha(Az, Az, p), F_\alpha(z, z, p), F_\alpha(z, Az, p), F_\alpha(z, A, p), F_\alpha(z, Az, p)\}$$

$$\Rightarrow \quad Az = z.$$

Thus z is common fixed point of A, S and T.

For **uniqueness**, let v ≠ z be another common fixed point of S, T and A.

By (1.1 a) we write

$$F_\alpha(Az, Av, kp) \geq \inf \left\{ \begin{matrix} F_\alpha(Tv, Av, p), F_\alpha(Sz, Az, p), F_\alpha(Sz, Tv, p), F_\alpha(Az, Tv, p), \\ F_\alpha(Sz, Av, p) \end{matrix} \right\}$$

$$F_\alpha(z, v, kp) \geq \inf \{F_\alpha(v, v, p), F_\alpha(z, z, p), F_\alpha(z, v, p), F_\alpha(z, v, p), F_\alpha(z, v, p)\}$$

Therefore **by lemma iii,** we write $z = v$. .

This completes the proof of Theorem (1.1)

Now we prove a result for rational expression.

Theorem (1.2): Let (X,F_α,t) be a complete Fuzzy Menger space, where t is continuous and t(p,p) ≥ p for all p and α in [0,1].Let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists continuous mappings A of X into S (X) ∩ T (X) which commute weakly with S and T and

(1.2a)

$$F_\alpha(Ax, Ay, kp) \geq \min \left\{ \begin{matrix} F_\alpha(Ty, Ay, p), F_\alpha(Sx, Ax, p), F_\alpha(Sx, Ty, p), \dfrac{F_\alpha(Sx, Ty, p)}{F_\alpha(Ax, Ty, p)}, \\ \dfrac{F_\alpha(Ty, Ay, p)}{F_\alpha(Sx, Ax, p))}, \dfrac{F_\alpha(Sx, Ax, p)}{F_\alpha(Ty, Ay, p)}, \end{matrix} \right\}$$

for all x,y ∈ X, t > 0,and 0 < q < 1. Then S, T and A have a unique common fixed point.

Proof: We define a sequence x_n such that

A,x_{2n}=S,x_{2n-1} and Ax_{2n-1} = T,x_{2n} n = 1, 2,--

We shall prove that {Ax_n} is a Cauchy sequence. By (1.2a), we have

$$F_\alpha(Ax_{2n}, Ax_{2n+1}, kp) \geq \min \left\{ \begin{matrix} F_\alpha(Tx_{2n+1}, Ax_{2n+1}, p), F_\alpha(Sx_{2n}, Ax_{2n}, p), F_\alpha(Sx_{2n}, Tx_{2n+1}, p), \\ \dfrac{F_\alpha(Sx_{2n}, Tx_{2n+1}, p)}{F_\alpha(Ax_{2n}, Tx_{2n+1}, p)}, \dfrac{F_\alpha(Tx_{2n+1}, Ax_{2n+1}, p)}{F_\alpha(Sx_{2n}, Ax_{2n}, p)}, \dfrac{F_\alpha(Sx_{2n}, Ax_{2n}, p)}{F_\alpha(Tx_{2n+1}, Ax_{2n+1}, p)} \end{matrix} \right\}$$

$$F_\alpha\left(Ax_{2n},Ax_{2n+1},kp\right)\geq\min\left\{\begin{array}{l}F_\alpha\left(Ax_{2n},Ax_{2n+1},p\right),F_\alpha\left(Ax_{2n+1},Ax_{2n},p\right),F_\alpha\left(Ax_{2n+1},Ax_{2n},p\right),\\ \dfrac{F_\alpha\left(Ax_{2n+1},Ax_{2n},p\right)}{F_\alpha\left(Ax_{2n},Ax_{2n},p\right)},\dfrac{F_\alpha\left(Ax_{2n},Ax_{2n+1},p\right)}{F_\alpha\left(Ax_{2n+1},Ax_{2n},p\right)},\dfrac{F_\alpha\left(Ax_{2n+1},Ax_{2n},p\right)}{F_\alpha\left(Ax_{2n},Ax_{2n+1},p\right)}\end{array}\right\}$$

$$=\min\left\{F_\alpha(Ax_{2n},Ax_{2n+1},p),F_\alpha(Ax_{2n+1},Ax_{2n},p),F_\alpha(Ax_{2n+1},Ax_{2n},p),\ 1,1,1\right\}$$

$$\geq\min\left\{F_\alpha(Ax_{2n-1},Ax_{2n},\frac{p}{k}),F_\alpha(Ax_{2n},Ax_{2n-1},\frac{p)}{k})\right\}$$

Therefore

$$F_\alpha\left(Ax_{2n},Ax_{2n+1},kp\right)\geq\ F_\alpha\left(Ax_{2n-1},Ax_{2n},\frac{p}{k}\right)$$

By induction

$$F_\alpha\left(Ax_{2k},Ax_{2m+1},kp\right)\geq F_\alpha\left(Ax_{2m},Ax_{2k-1},\frac{p}{k}\right)$$

For every k and m in N, Further if 2m +1 > 2k, then

$$F_\alpha\left(Ax_{2k},Ax_{2m+1},kp\right)\geq F_\alpha\left(Ax_{2k-1},Ax_{2m},\frac{p}{k}\right)\ldots\ldots.$$

$$\ldots\ldots\ldots.\geq F_\alpha\left(Ax_0,Ax_{2m+1-2k},\frac{p}{k^{2k}}\right)\text{ ----- (1.2b)}$$

If 2k > 2m+1, then

$$F_\alpha\left(Ax_{2k},Ax_{2m+1},kp\right)\geq F_\alpha\left(Ax_{2k-1},Ax_{2m},\frac{p}{k}\right)\ldots\ldots\ldots.\geq F_\alpha\left(Ax_{2k-(2m+1)},Ax_0,\frac{p}{k^{2m+1}}\right)\ldots\ldots.\ (1.2c)$$

By simple induction with (1.2b) and (1.2c)

We have

$$F_\alpha\left(Ax_n,Ax_{n+p},kp\right)\geq F_\alpha\left(Ax_0,Ax_p,\frac{p}{k^n}\right).$$

For n = 2k, p = 2m+1 or n = 2k+1, p = 2m +1

$$F_\alpha\left(Ax_n,Ax_{n+p},kp\right)$$
$$\geq F_\alpha\left(Ax_0,Ax_1,\frac{p}{2k^n}\right).F_\alpha\left(Ax_1,Ax_p,\frac{p}{k^n}\right).---(1.2d)$$

If n = 2k, p = 2m or n = 2k+1, p= 2m

For every positive integer p and n in N, by nothing that

$$F_\alpha\left(Ax_0,Ax_p,\frac{p}{k^n}\right)\to 1\,as\,n\to\infty$$

Thus { Ax_n } is a Cauchy sequence.

Since the space X is complete there exists z $\in$ X, such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n} = z$$

It follows that $Ax = Sz = Tz$ .Therefore

$$F_\alpha(Az, AAz, kp) \geq \min \left\{ \begin{array}{l} F_\alpha(TAz, AAz, p), F_\alpha(Sz, Az, p), F_\alpha(Sz, TAz, p), \\ \frac{F_\alpha(Sz, TAz, p)}{F_\alpha(Az, TAz, p)}, \frac{F_\alpha(TAz, AAz, p)}{F_\alpha(Sz, Az, p)}, \frac{F_\alpha(Sz, Az, p)}{F_\alpha(TAz, AAz, p)}, \end{array} \right\}$$

$$F_\alpha(Az, AAz, kp) \geq F_\alpha(Sz, TAz, p) \quad \geq F_\alpha(Sz, ATz, p) \geq F_\alpha(Az, AAz, p) .... \geq F_\alpha(Az, AAz, \frac{p}{k^n})$$

$$Since \lim_{n\to\infty} F_\alpha(Az, AAz, \frac{p}{k^n}) = 1 \Rightarrow Az = AAz$$

Thus z is common fixed point of A, S and T.

For uniqueness, let v((ξ)(v(ξ)≠ z(ξ)) be another common fixed point of S, T and A.

By (1.2a), we write

$$F_\alpha(Az, Av, kp) \geq \min \left\{ \begin{array}{l} F_\alpha(Tv, Av, p), F_\alpha(Sz, Az, p), F_\alpha(Sz, Tv, p), \\ \frac{F_\alpha(Sz, Tv, p)}{F_\alpha(Az, Tv, p)}, \frac{F_\alpha(Tv, Av, p)}{F_\alpha(Sz, Az, p)}, \frac{F_\alpha(Sz, Az, p)}{F_\alpha(Tv, Av, p)} \end{array} \right\}$$

$$F_\alpha(Az, Av, kp) \geq F_\alpha(z, v, p) \Rightarrow F_\alpha(z, v, kp) \geq F_\alpha(z, v, p)$$

Therefore by **lemma iii**, we write z(ξ)=v(ξ).

This completes the proof of Theorem (1.2).

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