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On Main Operators In Nonlinear Differential Games With Fixed Time

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Abstract.

In the present article we study some approximation properties of the main operators (upper and lower operators) and on the basis of these properties a connection between the upper and lower operators in nonlinear differential games and its applications to the problem of pursuit are established.

Keywords: differential games, admissible control, approximation, pursuer, evader, strategy.

1 Introduction

The structure of nonlinear differential games is described by operators T^t and $\prod^t$, [1,2] as natural generalization of the concept of alternating integral for linear differential games [3–5]. Publications [6-22] deal with further development of operator structures in nonlinear differential games. In particular, lower analogues operators T^t and $\prod^t$ and their applications to study of qualitative structure of phase space of differential games of pursuit-evasion were suggested in[10–13]. Questions of approximation of the operators T^t and $\prod^t$ by simpler operators were studied in [6,11]. In the future, for the symmetry operators T^t and $\prod^t$ are said to be the upper, and their lower analogues T_t and $\prod_t$ call lower operators in nonlinear differential games. In the present article we study some approximation properties of the operators $\prod^t$ and $\prod_t$ (upper and lower operators) and on the basis of these properties a connection between the upper and lower operators in nonlinear differential games and its applications to the problem of pursuit are established. Let $K(R^d)$ (respectively $C(R^d)$)be family of all nonempty compact (closed) subsets of R^d, $H = \{z \in R^d; |z| \leq 1\}$ be closed unit ball in R^d; $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n\}$ be partition of segment $[0, t]$ $(0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n = t$, n may depend on ω;).Let us assume r$\Delta_i = |\tau_i - \tau_{i-1}|$, and $|\omega| = t$, ω is a partition of the segment $[0, t]$. If X is subset of the Euclidean space, Δ is segment in R, then by $X[\Delta]$ denote the set of all measurable functions $a(\cdot) : \Delta \to X$. When $\Delta = [\alpha, \beta]$, we simply write $X[\alpha, \beta]$.

Let us consider the differential game

$$\frac{dz}{dt} = f(z, u, v), \qquad (1)$$

where $z \in R^d$, $u \in P$, $v \in Q$ and $f : R^d \times P \times Q \to R^d, P \in K(R^p), Q \in K(R^q)$. Along with the system (1) we also fix the set of M, $M \subset R^d$, which is called terminal set.

We suppose that further the function f holds the following conditions.

A. The function $f : R^d \times P \times Q \to R^d$ is continuous and is locally the Lipshitz type by z(i.e. the function f holds the Lipshitz condition on every compact set $D \in K(R^d)$ with the constant L_D, depending on compact D).

B. There is a constant $C \geq 0$, such that for all $z \in R^d$, $u \in P$, $v \in Q$, the inequality

$$| z \cdot f(z, u, v) | \leq C(1+ | z |^2)$$

holds.

C. The set $f(z, P, v)$ is convex for all $z \in R^d, v \in Q$, and the set $f(z, u, Q)$ is convex for all $z \in R^d, u \in P$.

We call every function $u(\cdot) \in P[\alpha, \beta]$ (respectively $v(\cdot) \in Q[\alpha, \beta]$) as admissible control of pursuer (respectively evader).We denote by $z(t, u(\cdot), v(\cdot), \xi)$ solution of the system (1),which corresponds to admissible control $u(t)$ and $v(t)$ and initial point $\xi \in R^d$(precisely definition of a trajectory is given in Section 4).Pursuit starts from a point $z_0 \in R^d \backslash M$ and it is considered to be ended, when the phase point hits the set M.In other words,pursuer aims to realize the inclusion $z(\tau) \in M$. Then, we say that pursuit from a point z_0 is completed at the time τ in the game (1). Naturally, there is a question: *From which initial points z_0 pursuit can be completed at the time τ in the game (1)?* To solve this problem L.S. Pontryagin has introduced the second method of pursuit in a linear differential game. The second method of pursuit is formulated in terms of alternating integral[3-5]. Solution of this problem for nonlinear differential games is described by operators T^t and $\prod^t$, [1,2] as natural generalization of the concept of alternating integral for linear differential games. In the present article, the basic definitions and results are presented in relation to operators $\prod^t$ and $\prod_t$.

Definition 1. The operator Π^ε associates every set $A \subset R^d$ with the set $\Pi^\varepsilon A$ of all points $\xi \in R^d$, such that for any admissible control of evader $v(\cdot) \in Q[0,\varepsilon]$ there is admissible control $u(\cdot) \in P[0,\varepsilon]$ of pursuer, such that the corresponding trajectory $z(t,u(\cdot),v(\cdot),\xi)$ with beginning at the initial point $\xi \in R^d$ hits $A \subset R^d$ at time ε, i e. $z(\varepsilon) \in A$.

Definition 2. The operator Π_ε associates every set $A \subset R^d$ with the set $\Pi_\varepsilon A$ of all points $\xi \in R^d$, such that there is admissible control pursuer $u(\cdot) \in P[0,\varepsilon]$ for any admissible control of evader $v(\cdot) \in Q[0,\varepsilon]$,moreover, corresponding trajectory $z(t,u(\cdot),v(\cdot),\xi)$ with the beginning at the point $\xi \in R^d$ hits $A \subset R^d$ at time ε, i e. $z(\varepsilon) \in A$.

By means of operations of association and intersection we can write operators Π^ε and Π_ε as follows:

$$\Pi^\varepsilon A = \bigcap_{v(\cdot)\in Q[0,\varepsilon]} \bigcup_{u(\cdot)\in P[0,\varepsilon]} \{\xi \in R^d \mid z(\varepsilon,u(\cdot),v(\cdot),\xi) \in A\}, \tag{2}$$

$$\Pi_\varepsilon A = \bigcup_{u(\cdot)\in P[0,\varepsilon]} \bigcap_{v(\cdot)\in Q[0,\varepsilon]} \{\xi \in R^d \mid z(\varepsilon,u(\cdot),v(\cdot),\xi) \in A\}. \tag{3}$$

Let $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n\}$ be partition of segment $[0,t]$. We assume

$$\Pi^\omega M = \Pi^{\delta_1}\Pi^{\delta_2}...\Pi^{\delta_n} M,$$

$$\Pi_\omega M = \Pi_{\delta_1}\Pi_{\delta_2}...\Pi_{\delta_n} M,$$

where $\delta_i = \tau_i - \tau_{i-1}, i = 1,2,...,n$.

Definition 3. Let $\Pi^t M = \bigcap\limits_{|\omega|=t} \Pi^\omega M$, $\Pi_t M = \bigcup\limits_{|\omega|=t} \Pi_\omega M$.

$\Pi^t M$ (respectively $\Pi_t M$) is called the upper (lower) operator of nonlinear differential games with fixed time [2-15].

Further, if it will be necessary, we shall indicate in notations the dependence of operators not only of ω or t, but also of other initial data, e. g. $\Pi_\varepsilon(M,P,Q)$, $\Pi^\varepsilon(M,P,Q)$.

A concepts of the upper and lower operators have the following role in nonlinear differential games: From points z_0 with $z_0 \in \Pi^t M$ (respectively $z_0 \in \Pi_t M$) the pursuit can be completed at the time τ with(without) discriminating against the evader controls [1-13].

2 Preliminaries

For completeness we state certain properties of the operators Π^ε and Π_ε. We note that for arbitrary family X_λ the following inclusions take place

$$\Pi^\varepsilon \bigcap_\lambda X_\lambda \subset \bigcap_\lambda \Pi^\varepsilon X_\lambda, \quad \bigcup_\lambda \Pi_\varepsilon X_\lambda \subset \Pi_\varepsilon \bigcup_\lambda X_\lambda. \tag{4}$$

Lemma 1 [6, 12]. Let $X_\lambda \in C(R^d)$ be non increasing (nondecreasing) direction of closed (open) sets. Then the equality $\Pi^\varepsilon \bigcap\limits_\lambda X_\lambda = \bigcap\limits_\lambda \Pi^\varepsilon X_\lambda$ $(\bigcup\limits_\lambda \Pi_\varepsilon X_\lambda = \Pi_\varepsilon \bigcup\limits_\lambda X_\lambda)$ is valid.

Lemma 2 [6, 12]. The following relations

a) $\Pi^{\varepsilon_1}\Pi^{\varepsilon_2}M \subset \Pi^{\varepsilon_1+\varepsilon_2}M,\ \ \Pi_{\varepsilon_1}\Pi_{\varepsilon_2}M \supset \Pi_{\varepsilon_1+\varepsilon_2}M.$

б)for any $|\omega_1| = t$, $|\omega_2| = t$ and $\omega_1 \subset \omega_2$, $\Pi^{\omega_2}M \subset \Pi^{\omega_1}M$,

$\Pi^{\omega_1}M \subset \Pi^{\omega_2}M$, are valid.

It were shown in[6,12] that 1) If $M \in C(R^d)$, then

$$\Pi^t M = \bigcap_{\delta>0} \Pi^t(M + \delta H);$$

2) if M is open subset of R^d , then

$$\Pi_t M = \bigcup_{\delta>0} \Pi_t(M \underline{*} \delta H).$$

Let operator A^ε (correspondingly A_ε) differs from the operator Π^ε (correspondingly from Π_ε) by the property that in Definitions 1 (correspondingly in Definition 2), only constant controls $v(\cdot) = v \in Q$ (correspondingly $u(\cdot) = u =\in P$) are taken instead of arbitrary admissible controls $v(\cdot) \in Q[0, \varepsilon]$ (correspondingly $u(\cdot) \in P[0, \varepsilon]$).

Let $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n\}$ be partition of segment $[0, t]$. We assume

$$A^\omega M = A^{\delta_1}A^{\delta_2}....A^{\delta_n}M,$$

$$A_\omega M = A_{\delta_1}A_{\delta_2}....A_{\delta_n}M,$$

where $\delta_i = \tau_i - \tau_{i-1}, i = 1, 2, ..., n.$

Definition 4. $A^t M = \bigcap\limits_{|\omega|=t} A^\omega M, A_t M = \bigcup\limits_{|\omega|=t} A_\omega M.$

Theorem 1 [6, 17]. We have the equality

$$\Pi^t M = A^t M \tag{5}$$

for $M \in C(R^d)$ and if M is open subset of R^d,then

$$\Pi_t M = A_t M. \tag{6}$$

Lemma 3 [6, 17]. Let ω_k be infinitely reducing sequence of partitions of the segment $[0, t]$, i. e., $\omega_k \subset \omega_{k+1}, |\ \omega_k\ | = t, \max\ |\ \tau_i^k - \tau_{i-1}^k\ | \to 0$ for $k \to \infty$. Then the following equality holds

$$\Pi^t M = \bigcap_{k\geq 1} \Pi^{\omega_k} M$$

for closed M and

$$\Pi_t M = \bigcup_{k\geq 1} \Pi_{\omega_k} M$$

for open M.

3 Approximation of the main operators

For nonlinear problems pursuit the construction of operators $\Pi^t M$ and $\Pi_t M$ is a lot of difficulties. Therefore the problem of working out effective schemes for the construction of these operators is relevant.

Consider the following operators accordingly

$$\Theta^{\varepsilon} B=\bigcap_{v\in Q}\bigcup_{u\in P}\{\xi\in R^d \mid z(\varepsilon,u,v,\xi)=\xi+\varepsilon f(\xi,u,v)\in B,\}$$

$$\Theta_{\varepsilon} B=\bigcup_{u\in P}\bigcap_{v\in Q}\{\xi\in R^d \mid z(\varepsilon,u,v,\xi)=\xi+\varepsilon f(\xi,u,v)\in B.\}$$

Definition operators Θ^t and Θ_t are similar to definition Π^t and Π_t respectively.

In this paper we consider the problem of approximation operators $\Pi^t M$ and $\Pi_t M$ by means of iteration of operators Θ^{ε} and Θ_{ε}, respectively. On the basis of these properties a connection between the operators $\Pi^t M$ and $\Pi_t M$ in nonlinear differential games and its applications to the problem of pursuit are established.

In what follows, we will assume that the boundary of M (∂M) is compact. We denote by D_* the set of all points of $\xi\in R^d$, of which it is possible to achieve the set ∂M (the boundary of M) at the appropriate admissible controls $u(\cdot)$ and $v(\cdot)$ for a time not exceeding θ. Let $D=D_*+H$ and constant is the quantity that can depend only on the function f, sets P,Q,D. We shall suppose that $t\le\theta$. Condition B guarantees boundedness of the set D [16]. We assume $K=\max\{| f(z,u,v) \| z\in D, u\in P, v\in Q\}$.

Lemma 4. There exists a positive number L such that the following inclusions

$$A^{\varepsilon}M\subset\Theta^{\varepsilon}(M+L\varepsilon^2H)\subset A^{\varepsilon}(M+2L\varepsilon^2H), \tag{7}$$

$$A_{\varepsilon}(M\underline{*}2L\varepsilon^2H)\subset\Theta_{\varepsilon}(M\underline{*}L\varepsilon^2H)\subset A_{\varepsilon}M \tag{8}$$

hold (see[4] about operators $+$ and $\underline{*}$).

The proofs are analogous, so we confine ourselves to the proof of (8). Let ξ be arbitrary element from the set $A_{\varepsilon}(M\underline{*}2L\varepsilon^2H)$. Then, there exists an admissible control of pursuer $u\in P$, such that for any admissible control evader $v(\cdot)\in Q[0,\varepsilon]$, the corresponding trajectory $z(t,u,v(\cdot),\xi)$ with the initial point of $\xi\in R^d$ hits $M\underline{*}2L\varepsilon^2H$ at time ε i.e. $z(\varepsilon)\in M\underline{*}2L\varepsilon^2H$. Therefore

$$z(\varepsilon,u,v(\cdot),\xi)=\xi+\int_0^{\varepsilon} f(z(t),u,v(t))dt\in M\underline{*}2L\varepsilon^2H. \tag{9}$$

In virtue of the condition A for arbitrary controls $u\in P$, $v(\cdot)\in Q$ and the initial point $\xi\in R^d$ we have the relation

$$\mid f(z(t),u,v(t))-f(\xi,u,v(t))\mid\le L_1\mid z(t)-\xi\mid. \tag{10}$$

On the other hand,

$$\mid z(t,u,v(t),\xi)-\xi\ \mid\le K\varepsilon, t\in[0,\varepsilon]. \tag{11}$$

Hence, using the inequality (10), we obtain

$$\mid f(z(t),u,v(t))-f(\xi,u,v(t))\mid\le L\varepsilon, \tag{12}$$

where $L=L_1K$.

Let us prove now that for every $v(\cdot)\in Q[0,\varepsilon]$, there is constant control $v\in Q$, for which the equality

$$\xi+\int_0^{\varepsilon} f(\xi,u,v(t))dt=\xi+\varepsilon f(\xi,u,v), \tag{13}$$

take place.

Due to the condition C, the set $f(\xi,u,Q)$ is convex for every $u\in P$.Therefore we have

$$\int_0^{\varepsilon} f(\xi,u,v(t))dt\in\varepsilon f(\xi,u,Q).$$

It follows that there is a $v\in Q$ such that

$$\int_0^{\varepsilon} f(\xi,u,v(t))dt=\varepsilon f(\xi,u,v).$$

Consequently, for any $v(\cdot)\in Q[0,\varepsilon]$ there is a constant control $v\in Q$, for which the equality

$$\xi+\int_0^{\varepsilon} f(\xi,u,v(t))dt=\xi+\varepsilon f(\xi,u,v)$$

holds. Applying inequality (12) to the right side of the equality (13), we obtain

$$\xi+\varepsilon f(\xi,u,v)\in\xi+\int_0^{\varepsilon} f(z(t),u,v(t))dt+L\varepsilon^2 H. \tag{14}$$

Using the inclusion (9), we have

$$\xi+\varepsilon f(\xi,u,v)\in M\underline{*}2L\varepsilon^2 H+L\varepsilon^2 H.$$

Hence,

$$\xi\in\Theta_{\varepsilon}(M\underline{*}L\varepsilon^2 H).$$

Similarly, the right side of the turn proved (8). Proof of the inclusion (7) is similar to proof of the relation (8).

Lemma 5. The following inclusions are valid

$$\Theta^{\varepsilon}(M)+L\delta^2 H\subset\Theta^{\varepsilon}(M+L\delta^2(1+L_1\varepsilon)H). \tag{15}$$

$$\Theta_{\varepsilon}(M\underline{*}L\delta^2(1+L_1\varepsilon)H)+L\delta^2 H\subset\Theta_{\varepsilon}M. \tag{16}$$

Proof. Let η be an arbitrary element of the left side inclusion (15). Then there is $\xi\in\Theta^{\varepsilon}(M)$ such that

$$\mid\eta-\xi\mid\leq L\delta^2. \tag{17}$$

By virtue of the condition A we have

$$\mid f(\xi,u,v)-f(\eta,u,v)\mid\leq L_1\mid\eta-\xi\mid. \tag{18}$$

Now,using (17), we obtain

$$\mid f(\xi,u,v)-f(\eta,u,v)\mid\leq L_1 L\delta^2. \tag{19}$$

Consider the sum of $\eta+\varepsilon f(\eta,u,v)$. Using inequality (17) and (18) we have

$$\eta+\varepsilon f(\eta,u,v)\in\xi+L\delta^2 H+\varepsilon(f(\xi,u,v)+L_1L\delta^2 H)=\xi+\varepsilon f(\xi,u,v)+L\delta^2(1+L_1\varepsilon).$$

Now, by virtue of condition $\xi\in\Theta^{\varepsilon}(M)$ we have $\eta+\varepsilon f(\eta,u,v)\in A+L\Delta^2(1+L_1\varepsilon)$.

Hence, $\eta\in\Theta^{\varepsilon}(A+L\delta^2(1+L_1\varepsilon)H)$. Proof of inclusion (16) is similar to proof of the relation (15). Lemma 5 is proved.

Further, we consider only uniform partition of the segment $[0,t]$. Let $\omega_n=\{0,\varepsilon,2\varepsilon,...,n\varepsilon=t\}$ be uniform partition of the segment $[0,t]$, where $\varepsilon=\frac{t}{n}$. Let $\Gamma(\varepsilon)=L\varepsilon^2\sum_{K=1}^{n}(1+L_1\varepsilon)^{k-1}$. We assume

$$\Theta^{2\varepsilon}M=\Theta^{\varepsilon}\Theta^{\varepsilon}M,\Theta^{k\varepsilon}M=\Theta^{\varepsilon}\Theta^{(k-1)\varepsilon}M,\Theta^{\omega_n}M=\Theta^{n\varepsilon}M,$$

$$\Theta_{2\varepsilon}M=\Theta_{\varepsilon}\Theta_{\varepsilon}M,\Theta_{k\varepsilon}M=\Theta_{\varepsilon}\Theta_{(k-1)\varepsilon}M,\Theta_{\omega_n}M=\Theta_{n\varepsilon}M.$$

Note that for convenience entry is similar to the notation $\Theta^{k\varepsilon}, \Theta_{k\varepsilon}$ introduced $A^{k\varepsilon}, A_{k\varepsilon}$.

Theorem 3. The following inclusions

$$A^{\omega_n}M\subset\Theta^{\omega_n}(M+\Gamma(\varepsilon)H)\subset A^{\omega_n}(M+2\Gamma(\varepsilon)H), \tag{20}$$

$$A_{\omega_n}(M\underline{*}2\Gamma(\varepsilon)H)\subset\Theta_{\omega_n}(M\underline{*}\Gamma(\varepsilon)H)\subset A_{\omega_n}(M) \tag{21}$$

are valid.

Proof. From Lemma 4 it follows that

$$A^{\varepsilon}M\subset\Theta^{\varepsilon}(M+L\varepsilon^2H).$$

Using Lemma 4 again we obtain

$$A^{2\varepsilon}M\subset\Theta^{\varepsilon}(\Theta^{\varepsilon}(M+L\varepsilon^2H)+L\varepsilon^2H).$$

Applying Lemma 5 to the right side of this inclusion we have

$$A^{2\varepsilon}M\subset\Theta^{2\varepsilon}(M+L\varepsilon^2(1+L_1\varepsilon)H).$$

Suppose

$$A^{p\varepsilon}M\subset\Theta^{p\varepsilon}(M+L\varepsilon^2\sum_{k=1}^{p}(1+L_1\varepsilon)^{k-1}H). \tag{22}$$

We shall show that

$$A^{(p+1)\varepsilon}M\subset\Theta^{(p+1)\varepsilon}(M+L\varepsilon^2\sum_{k=1}^{p+1}(1+L_1\varepsilon)^{k-1}H). \tag{23}$$

Applying Lemma 4 to the inclusion (21), we obtain

$$A^{(p+1)\varepsilon}M\subset\Theta^{\varepsilon}(\Theta^{p\varepsilon}(M+L\varepsilon^2\sum_{k=1}^{p}(1+L_1\varepsilon)^{k-1}H)+L\varepsilon^2).$$

Now, using Lemma 5 we have at the relation r

$$A^{(p+1)\varepsilon}M\subset\Theta^{(p+1)\varepsilon}(M+L\varepsilon^2\sum_{k=1}^{p+1}(1+L_1\varepsilon)^{k-1}H).$$

This implies

$$A^{n\varepsilon}M\subset\Theta^{n\varepsilon}(M+L\varepsilon^2\sum_{k=1}^{n}(1+L_1\varepsilon)^{k-1}H).$$

Consequently,

$$A^{\omega_n}M\subset\Theta^{\omega_n}(M+\Gamma(\varepsilon)H).$$

Similarly of that, the following inclusion

$$\Theta^{\omega_n}(M+\Gamma(\varepsilon)H)\subset A^{\omega_n}(M+2\Gamma(\varepsilon)H)$$

will established.

Consequently,

$$A^{\omega_n}M\subset\Theta^{\omega_n}(M+\Gamma(\varepsilon)H)\subset A^{\omega_n}(M+2\Gamma(\varepsilon)H).$$

Theorem 3 is proved.

Theorem 4. The following equality holds

$$\Pi^t M = \bigcup_{\delta>0} \Theta^t(M + \delta H), \tag{24}$$

for $M \in C(R^d)$,

$$\Pi_t M = \bigcup_{\delta>0} \Theta_t(M \underline{*} \delta H), \tag{25}$$

for open M, $M \subset R^d$.

Proof. Consider the value of $\Gamma(\varepsilon) = L\varepsilon^2 \sum_{k=1}^{n} (1 + L_1\varepsilon)^k$. It is easy to see that $\Gamma(\varepsilon) \leq \varepsilon L(e^{L_1\theta} - 1)$. By choose of partitions we have $\Gamma(\varepsilon) \leq \varepsilon L(e^{L\theta} - 1) < \delta$, i.e. $\varepsilon < \frac{\delta}{L(e^{L_1\theta}-1)}$. Inclusion (19) implies

$$A^{\omega_n} M \subset \Theta^{\omega_n}(M + \delta H) \subset A^{\omega_n}(M + 2\delta H).$$

Transition to the intersection on the ω_n in these relations term by term, we have

$$A^t M \subset \bigcap_{\omega_n} \Theta^{\omega}(M + \delta H) \subset \bigcup_{\omega} A^{\omega}(M + 2\delta H).$$

Now taking into account Lemma 1 we obtain,

$$A^t M \subset \Theta^t(M + \delta H) \subset A^t(M + 2\delta H).$$

Since $\delta > 0$ is arbitrary we have

$$A^t M \subset \bigcap_{\delta>0} \Theta^t(M + \delta H) \subset \bigcap_{\delta>0} A^t(M + 2\delta H).$$

Now, Theorem 1 and Theorem 2 imply

$$\Pi^t M = \bigcap_{\delta>0} \Theta^t(M + \delta H).$$

The proof of the equality (24) is similar to the proof of (23). Theorem 4 is proved.

4 The connection between the upper and lower operators and its application to the nonlinear differential games of pursuit with fixed time

We shall study connection between operators Π^t and Π_t.

Theorem 5. For any $\delta > 0$, there exists a number ε, for all $\mid \omega_n \mid < \varepsilon$ the inclusions are valid

$$\Theta^{\omega_n} M \subset \Theta_{\omega_n}(M + \delta H), \tag{26}$$

$$\Theta^{\omega_n}(M \underline{*} \delta H) \subset \Theta_{\omega_n} M. \tag{27}$$

The proofs are analogous, so we confine ourselves to the proof of (25). For convenience, we consider the operator $\Theta^{3\varepsilon}$. By definition

$$\Theta^{3\varepsilon} M = \bigcap_{v \in Q} \bigcup_{u \in P} \{\xi_1 \in R^d \mid z(\varepsilon, u, v, \xi_1) = \xi_1 + \varepsilon f(\xi_1, u, v) \in \Theta^{2\varepsilon} M.\}$$

Furthermore,

$$\Theta^{3\varepsilon}M \subset \bigcup_{u\in P}\bigcap_{v\in Q}\{\xi_1 \in R^d \mid \xi_1 + \varepsilon f(\xi_1,u,v) \in \bigcup_{u\in P}[\Theta^{2\varepsilon}M - \varepsilon f(\xi_1,u,v)] + K\varepsilon H\}.$$

Here we considered the fact $\varepsilon f(\xi_1,u,v) \subset K\varepsilon H$.

By repeating this process with respect to $\Theta^{2\varepsilon}$ and Θ^{ε} we get

$$\Theta^{3\varepsilon}M \subset \bigcup_{u\in P}\bigcap_{v\in Q}\{\xi_1 \in R^d \mid \xi_1 + \varepsilon f(\xi_1,u,v) \in \bigcup_{u\in P}\bigcap_{v\in Q}\{\xi_2 \in R^d \mid \xi_2 \in$$

$$\in \bigcup_{u\in P}[\bigcap_{v\in Q}[\xi_3 \in R^d \mid \xi_3 \in \bigcup_{u\in P}[M - \varepsilon f(\xi_3,u,v)] - \varepsilon f(\xi_2,u,v)] - \varepsilon f(\xi_1,u,v)] + K\varepsilon H\}. \quad (28)$$

Note that $\xi_{i+1} = \xi_i + \varepsilon f(\xi_i,u,v)$. Therefore, in virtue of the condition A, we have $\mid f(\xi_{i+1},u,v) - f(\xi_i,u,v) \mid \leq L_1 \mid \xi_{i+1} - \xi_i \mid \leq L_1 K\varepsilon H = L\varepsilon H$, where $L = L_1K$. This implies $\varepsilon f(\xi_{i+1},u,v) \subset \varepsilon f(\xi_i,u,v) + L\varepsilon^2, \quad \varepsilon f(\xi_i,u,v) \subset \varepsilon f(\xi_{i+1},u,v) + L\varepsilon^2$.

Taking into account of these relations,we replace any vector $\varepsilon f(\xi_i,u,v)$ in the right side of the inclusions (27) on $\varepsilon f(\xi_{i+1},u,v) + L\varepsilon^2$ for $i = 1,2$ and using the inclusion (4) we obtain

$$\Theta^{3\varepsilon}M \subset \bigcup_{u\ inP}\bigcap_{v\in Q}\{\xi_1 \in R^d \mid \xi_1 + \varepsilon f(\xi_1,u,v) \in \bigcup_{u\in P}\bigcap_{v\in Q}[\xi_2 \in R^d \mid \xi_2 \in$$

$$\in \bigcup_{u\in P}\bigcap_{v\in Q}[\xi_3 \in R^d \mid \xi_3 \in [M - \varepsilon f(\xi_3,u,v) + L\varepsilon^2 + K\varepsilon H]] - \varepsilon f(\xi 2,u,v)+$$

$$+L\varepsilon^2] + K\varepsilon H\}.$$

This implies,

$$\Theta^{3\varepsilon}M \subset \bigcup_{u\in P}\bigcap_{v\in Q}\{\xi_1 \in R^d \mid \xi_1 + \varepsilon f(\xi_1,u,v) \in \bigcup_{u\in P}\bigcap_{v\in Q}[\xi_2 \in R^d \mid \xi_2 + \varepsilon f(\xi_2,u,v) \in$$

$$\in \bigcup_{u\in P}\bigcap_{v\in Q}[\xi_3 \in R^d \mid \xi_3 + \varepsilon f(\xi_3,u,v) \in [M + L\varepsilon^2 + K\varepsilon H]] + L\varepsilon^2] + K\varepsilon H\}.$$

Consequently,

$$\Theta^{3\varepsilon}M \subset \Theta_\varepsilon(\Theta_\varepsilon(\Theta_\varepsilon(M + L\varepsilon^2H + K\varepsilon H) + L\varepsilon^2H) + K\varepsilon H). \quad (29)$$

Now, applying Lemma 5 to the right side of the inclusion (28) we have

$$\Theta^{3\varepsilon}M \subset \Theta_{3\varepsilon}(M + L\varepsilon^2H + K\varepsilon H + L\varepsilon^2(1 + L_1\varepsilon)H + K\varepsilon(1 + L_1\varepsilon)^2H)$$

i.e.

$$\Theta^{3\varepsilon}M \subset \Theta_{3\varepsilon}(M + (L\varepsilon^2\sum_{k=1}^{2}(1 + L_1\varepsilon)^{k-1} + K\varepsilon(1 + (1 + L_1\varepsilon)^2))H)$$

.

Repeating this process one more time we obtain

$$\Theta^{n\varepsilon}M \subset \Theta_{n\varepsilon}(M + \Gamma(\varepsilon)H), \quad (30)$$

where

$$\Gamma(\varepsilon) = L\varepsilon^2\sum_{k=1}^{n-1}(1 + L_1\varepsilon)^{k-1} + K\varepsilon(1 + (1 + L_1\varepsilon)^{n-1}).$$

To complete the proof of Theorem 5 it suffices to choose $\varepsilon > 0$ satisfying the inequality $\Gamma(\varepsilon) < \delta$, as it was done in the proof of Theorem 4.

Therefore, for all $\mid \omega_n \mid < \varepsilon$ the inclusion

$$\Theta^{\omega_n} M \subset \Theta_{\omega_n}(M + \delta H)$$

holds. Theorem 5 is proved.

Theorem 6. The following equality holds

$$\Pi^t M = \bigcap_{\delta > 0} \Pi_t(M + \delta H), \tag{31}$$

for $M \in C(R^d)$ and

$$\Pi_t M = \bigcup_{\delta > 0} \Pi^t(M \underline{*} \delta H) \tag{32}$$

for open M, $M \subset R^d$.

Proof. From Theorem 5 it follows that

$$\Pi^{\omega_n} M \subset \Theta^{\omega_n}(M + \delta H) \subset \Theta_{\omega_n}(M + 2\delta H) \subset$$

$$\subset \Pi_{\omega_n}(M + 3\delta H).$$

This implies that for any $\delta > 0$, there exists a number $\varepsilon > 0$ for all ω_n, such that $\mid \omega_n \mid < \varepsilon$ the relation

$$\Pi^{\omega_n} M \subset \Pi_{\omega_n}(M + 3\delta H).$$

Hence, by Lemma 3, we have

$$\Pi^t M) \subset \bigcap_{\delta > 0} \Pi_t(M + 3\delta H).$$

On the other hand,

$$\bigcap_{\delta > 0} \Pi_t(M + 3\delta H) \subset \bigcap_{\delta > 0} \Pi^t(M + 3\delta H).$$

Now, using the first part of Theorem 1, we have

$$\bigcap_{\delta > 0} \Pi_t(M + 3\delta H) \subset \Pi^t M.$$

Hence, we obtain the equality (30).The proof of the equality (31) is similar to the proof of (30).

Theorem 6 is proved.

Applications of upper and lower operators to nonlinear differential games is similarly to the linear case[2,4,10-13,19]. Therefore, we confine ourselves to a brief presentation of the definitions of basic concepts and to state basic results in connection with the system(1).

In further for brevity entry we assume $P(\varepsilon) = P[0, \varepsilon]$ and $Q(\varepsilon) = Q[0, \varepsilon]$.

Defiition 5. The mapping $V_\varepsilon^* : R^d \to Q(\varepsilon)$ is said to be ε- strategy of the evader in the upper game.The mapping $U_\varepsilon^* : R^d \times Q(\varepsilon) \to P(\varepsilon)$ is said to be ε-strategy of the pursuer in the upper game.

Defiition 6. The mapping $U_*^\varepsilon : R^d \to P(\varepsilon)$ is said to be ε- strategy of the pursuer in the lower game.The mapping $V_*^\varepsilon : R^d \times P(\varepsilon) \to Q(\varepsilon)$ is said to be ε-strategy of the evader in the lower game.

A given initial point z_0 and a given pair of strategies U_ε^*, V_ε^* give rise to a unique trajectory $z(t) = z(t, z_0, U_\varepsilon^*, V_\varepsilon^*)$, $t \geq 0$. This trajectory is defined on $[0, \varepsilon]$ as solution of the Cauchy problem

$$\frac{dz}{dt} = f(z(t), u_0(t), v_0(t)), z(0) = z_0,$$

where $v_0(\cdot) = V_\varepsilon^*(z_0)$ and $u_0(\cdot) = U_\varepsilon^*(z_0, v_0(\cdot))$

The trajectory is then extended from $[0, k\varepsilon]$ to $[0, (k+1)\varepsilon]$ as the solution Cauchy problem

$$\frac{dz}{dt} = f(z(t), u_k(t), v_k(t)), z(k\varepsilon + 0) = z(k\varepsilon - 0),$$

where $v_k(t) = v(t - k\varepsilon), v(\cdot) = V_\varepsilon^*(z(k\varepsilon))$ and $u_k(t) = U_\varepsilon^*(z_{k\varepsilon}, v(\cdot))(t - k\varepsilon), t \in [k\varepsilon, (k+1)\varepsilon]$.

The trajectory $z(t) = z(t, z_0, U_*^\varepsilon, V_*^\varepsilon)$ corresponding to a given initial point z_0 and a given pair of strategies U_*^ε, V_*^ε is defined similarly.

Defiition 7. Pursuit from the point z_0 can be completed at the time τ in the upper game if for any $\varepsilon = \frac{\tau}{n}$, there exists ε-strategy of the pursuer U_ε^* such that $z(\tau) = z(\tau, z_0, U_\varepsilon^*, V_\varepsilon^*) \in M$ for any ε-strategy of the evader V_ε^*.

Concept of possibility to complete pursuit at the time τ in the lower game can be introduced similarly.

On the base of these definitions, the equality (30) in Theorem 6 can be interpreted as follows. If $M \in C(R^d)$, then pursuit from a point z_0 can be completed at the time τ in the upper game if and only if the pursuer in the lower game can transfer the phase point from the initial point z_0 into any neighborhood of the terminal set at the time τ(see [2,3,10,12,21-22])

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