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How to prove the Riemann Hypothesis

Fayez Fok Al Adeh[1].
[1]President of the Syrian Cosmological Society
P.O.Box:13187, Damascus, SyriaTels:00963-11-2776729,2713005
*Corresponding Author Email address: hayfa@scs-net.org

ABSTRACT

The aim of this paper is to prove the celebrated Riemann Hypothesis.
I have already discovered a simple proof of the Riemann Hypothesis. The hypothesis states that the nontrivial zeros of the Riemann zeta function have real part equal to 0.5. I assume that any such zero is s = a+ bi. I use integral calculus in the first part of the proof. In the second part I employ variational calculus. Through equations (50) to (59) I consider (a) as a fixed exponent, and verify that a = 0.5 .From equation (60) onward I view (a) as a parameter (a <0.5) and arrive at a contradiction. At the end of the proof (from equation (73)) and through the assumption that (a) is a parameter, I verify again that a = 0.5

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INTRODUCTION

The Riemann zeta function is the function of the complex variable s = a + bi (i = $\sqrt{-1}$), defined in the half plane a >1 by the absolute convergent series

(1) $$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s}$$

and in the whole complex plane by analytic continuation.

The function $\zeta(s)$ has zeros at the negative even integers -2, -4, … and one refers to them as the trivial zeros. The Riemann hypothesis states that the nontrivial zeros of $\zeta(s)$ have real part equal to 0.5.

PROOF OF THE HYPOTHESIS

We begin with the equation

(2) $\zeta(s)=0$

And with

(3) s= a+bi

(4) $\zeta(a+bi)=0$

It is known that the nontrivial zeros of $\zeta(s)$ are all complex. Their real parts lie between zero and one.

If $0 < a < 1$ then

(5) $$\zeta(s) = s\int_0^\infty \frac{[x]-x}{x^{s+1}}\,dx \qquad (0 < a < 1)$$

[x] is the integer function

Hence

(6) $$\int_0^\infty \frac{[x]-x}{x^{s+1}}\,dx = 0$$

Therefore

(7) $$\int_0^\infty ([x]-x)x^{-1-a-bi}dx = 0$$

(8) $$\int_0^\infty ([x]-x)x^{-1-a}x^{-bi}dx = 0$$

(9) $$\int_0^\infty x^{-1-a}([x]-x)(\cos(b\log x) - i\sin(b\log x))dx = 0$$

Separating the real and imaginary parts we get

(10) $$\int_0^\infty x^{-1-a}([x]-x)\cos(b\log x)dx = 0$$

(11) $$\int_0^\infty x^{-1-a}([x]-x)\sin(b\log x)dx = 0$$

According to the functional equation, if $\zeta(s)$=0 then $\zeta(1-s)$=0. Hence we get besides equation (11)

(12) $$\int_0^\infty x^{-2+a}([x]-x)\sin(b\log x)dx = 0$$

In equation (11) replace the dummy variable x by the dummy variable y

(13) $$\int_0^\infty y^{-1-a}([y]-y)\sin(b\log y)dy = 0$$

We form the product of the integrals (12)and (13).This is justified by the fact that both integrals (12) and (13) are absolutely convergent .As to integral (12) we notice that

$$\int_0^\infty x^{-2+a}([x]-x)\sin(b\log x)dx \le \int_0^\infty \left| x^{-2+a}([x]-x)\sin(b\log x)\right| dx$$

$$\leq \int_0^\infty x^{-2+a}((x))dx$$

(where ((z)) is the fractional part of z , $0\leq$ ((z))<1)

$$= \lim(t\to 0) \int_0^{1-t} x^{-1+a}dx + \lim(t\to 0) \int_{1+t}^\infty x^{-2+a}((x))dx$$

(t is a very small positive number) (since ((x)) =x whenever $0\leq x<1$)

$$= \frac{1}{a} + \lim(t\to 0) \int_{1+t}^\infty x^{-2+a}((x))dx$$

$$< \frac{1}{a} + \lim(t\to 0) \int_{1+t}^\infty x^{-2+a}dx = \frac{1}{a} + \frac{1}{a-1}$$

And as to integral (13) $\int_0^\infty y^{-1-a}([y]-y)\sin(b\log y)dy$

$$\leq \int_0^\infty \left| y^{-1-a}([y]-y)\sin(b\log y) \right| dy$$

$$\leq \int_0^\infty y^{-1-a}((y))dy$$

$$= \lim(t\to 0) \int_0^{1-t} y^{-a}dy + \lim(t\to 0) \int_{1+t}^\infty y^{-1-a}((y))dy$$

(t is a very small positive number) (since ((y)) =y whenever $0\leq y<1$)

$$= \frac{1}{1-a} + \lim(t\to 0) \int_{1+t}^\infty y^{-1-a}((y))dy$$

$$< \frac{1}{1-a} + \int_{1+t}^\infty y^{-1-a}dy = \frac{1}{1-a} + \frac{1}{a}$$

Since the limits of integration do not involve x or y , the product can be expressed as the double integral

(14) $$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)\sin(b\log y)\sin(b\log x)dxdy = 0$$

Thus

(15)

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)(\cos(b\log y + b\log x) - \cos(b\log y - b\log x))dxdy = 0$$

(16) $$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)(\cos(b\log xy)-\cos(b\log\frac{y}{x}))dxdy=0$$

That is

(17) $$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)\cos(b\log xy)dxdy=$$

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)\cos(b\log\frac{y}{x})dxdy$$

Consider the integral on the right-hand side of equation (17)

(18) $$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)\cos(b\log\frac{y}{x})dxdy$$

In this integral make the substitution x = $\frac{1}{z}$ dx = $\frac{-dz}{z^2}$

The integral becomes

(19) $$\int_0^\infty\int_\infty^0 z^{2-a}y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)\frac{-dz}{z^2}dy$$

That is

(20) $$-\int_0^\infty\int_\infty^0 z^{-a}y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)dzdy$$

This is equivalent to

(21) $$\int_0^\infty\int_0^\infty z^{-a}y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)dzdy$$

If we replace the dummy variable z by the dummy variable x, the integral takes the form

(22) $$\int_0^\infty\int_0^\infty x^{-a}y^{-1-a}([\frac{1}{x}]-\frac{1}{x})([y]-y)\cos(b\log xy)dxdy$$

Rewrite this integral in the equivalent form

(23) $$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})([y]-y)\cos(b\log xy)dxdy$$

Thus equation 17 becomes

(24)

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)\cos(b\log xy)dxdy=$$

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})([y]-y)\cos(b\log xy)dxdy$$

Write the last equation in the form

(25) $\int_0^\infty\int_0^\infty x^{-2+a} y^{-1-a}([y]-y)\cos(b\log xy)\{(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})-([x]-x)\}dx\,dy=0$

Let p >0 be an arbitrary small positive number.We consider the following regions in the x –y plane.

(26) The region of integration I = $[0,\infty)\times[0,\infty)$

(27) The large region I1 =$[p,\infty)\times[p,\infty)$

(28) The narrow strip I 2 =$[p,\infty)\times[0,p]$

(29) The narrow strip I 3 = $[0,p]\times[0,\infty)$

Note that

(30) I = I1 $\bigcup$ I 2 $\bigcup$ I 3

Denote the integrand in the left hand side of equation (25) by

(31) F (x,y) = $x^{-2+a} y^{-1-a}([y]-y)\cos(b\log xy)\{(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})-([x]-x)\}$

Let us find the limit of F (x,y) as x $\to\infty$ and y $\to\infty$. This limit is given by

(32) Lim $x^{-a} y^{-1-a}$[- ((y))] cos (blog xy) [- (($\frac{1}{x}$)) + ((x)) x^{2a-2}]

((z)) is the fractional part of the number z ,0$\le$ ((z)) < 1

The above limit vanishes ,since all the functions [- ((y))] ,cos (blog xy), - (($\frac{1}{x}$)), and ((x)) remain bounded as x $\to\infty$ and y $\to\infty$

Note that the function F (x,y) is defined and bounded in the region I 1. We can prove that the integral

(33) $\iint_{I1}$ F(x,y) dx dy is bounded as follows

(34) $\iint_{I1}$ F(x,y) dx dy = $\iint_{I1} x^{-a} y^{-1-a}$[- ((y))] cos (blog xy) [- (($\frac{1}{x}$)) + ((x)) x^{2a-2}] dx dy

$\le \left| \iint_{I1} x^{-a} y^{-1-a}\right.$[- ((y))] cos (blog xy) [- (($\frac{1}{x}$)) + ((x)) x^{2a-2}] dx dy $\left.\right|$

$$= \left| \int_p^{\infty} \left(\int_p^{\infty} x^{-a} \cos(b\log xy)\left[-\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx \right) y^{-1-a} [-((y))] dy \right|$$

$$\leq \int_p^{\infty} \left| \left(\int_p^{\infty} x^{-a} \cos(b\log xy)\left[-\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx \right) \right| \left| y^{-1-a} [-((y))] \right| dy$$

$$\leq \int_p^{\infty} \left(\int_p^{\infty} x^{-a} \left| \cos(b\log xy) \right| \left| \left[-\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] \right| dx \right) \left| y^{-1-a} [-((y))] \right| dy$$

$$< \int_P^{\infty} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx \int_P^{\infty} y^{-1-a}$$

$$= \frac{1}{ap^a} \int_P^{\infty} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx$$

$$= \frac{1}{ap^a} \left\{ \lim(t\to 0) \int_P^{1-t} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx + \lim(t\to 0) \right.$$

$$\left. \int_{1+t}^{\infty} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx \right\}$$

where t is a very small arbitrary positive. number. Since the integral

$\lim(t\to 0) \int_P^{1-t} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx$ is bounded, it remains to

show that $\lim(t\to 0)$

$\int_{1+t}^{\infty} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right] dx$ is bounded.

Since x >1 ,then $\left(\left(\frac{1}{x}\right)\right) = \frac{1}{x}$ and we have $\lim(t\to 0) \int_{1+t}^{\infty} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x))\, x^{2a-2} \right.$

$\left.\right] dx$

$$= \lim(t\to 0) \int_{1+t}^{\infty} x^{-a} \left[\frac{1}{x} + ((x))\, x^{2a-2} \right] dx$$

$$= \lim(t\to 0) \int_{1+t}^{\infty} \left[x^{-a-1} + ((x))\, x^{a-2} \right] dx$$

$$< \lim(t\to 0) \int_{1+t}^{\infty} \left[x^{-a-1} + x^{a-2} \right] dx$$

$$= \frac{1}{a(1-a)}$$

Hence the boundedness of the integral $\iint$ F(x,y) dx dy is proved.

I1

Consider the region

(35) I4=I2 $\cup$ I3

We know that

(36) $0 = \iint_{I} F(x,y)\,dx\,dy = \iint_{I1} F(x,y)\,dx\,dy + \iint_{I4} F(x,y)\,dx\,dy$

and that

(37) $\iint_{I1} F(x,y)\,dx\,dy$ is bounded

From which we deduce that the integral

(38) $\iint_{I4} F(x,y)\,dx\,dy$ is bounded

Remember that

(39) $\iint_{I4} F(x,y)\,dx\,dy = \iint_{I2} F(x,y)\,dx\,dy + \iint_{I3} F(x,y)\,dx\,dy$

Consider the integral

(40) $\iint_{I2} F(x,y)\,dx\,dy \le \left| \iint_{I2} F(x,y)\,dx\,dy \right|$

$$= \left| \int_0^p \left(\int_p^\infty x^{-a} \{ ((\frac{1}{x})) - ((x))\ x^{2a-2} \} \cos(b \log xy)\,dx \right) \frac{1}{y^a}\,dy \right|$$

$$\le \int_0^p \left| \int_p^\infty \left(x^{-a} \{ ((\frac{1}{x})) - ((x))\ x^{2a-2} \} \cos(b \log xy)\,dx \right) \right| \frac{1}{y^a}\,dy$$

$$\le \int_0^p \left(\int_p^\infty \left| x^{-a} \{ ((\frac{1}{x})) - ((x))\ x^{2a-2} \} \right| \left| \cos(b \log xy) \right| dx \right) \frac{1}{y^a}\,dy$$

$$\le \int_P^\infty \left| x^{-a} \{ ((\frac{1}{x})) - ((x))\ x^{2a-2} \} \right| dx \times \int_0^P \frac{1}{y^a}\,dy$$

(This is because in this region ((y)) = y). It is evident that the integral $\int_P^{\infty} \left| x^{-a}\{((\frac{1}{x}))-((x))\, x^{2a-2}\}\right|$ dx is bounded, this was proved in the course of proving that the integral $\iint_{I1}$ F(x,y) dx dy is bounded .Also it is evident that the integral

$$\int_0^P \frac{1}{y^a}\,dy$$

is bounded. Thus we deduce that the integral (40) $\iint_{I2}$ F(x,y) dx dy is bounded

Hence, according to equation(39),the integral $\iint_{I3}$ F(x,y) dx dy is bounded.

Now consider the integral

(41) $\iint_{I3}$ F(x,y) dx dy

We write it in the form

(42) $$\iint_{I3} F(x,y)\,dx\,dy = \int_0^p \left(\int_0^{\infty} y^{-1-a}\,((y))\cos(b\log xy)\,dy\right)\frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}\,dx$$

(This is because in this region ((x)) = x)

$$\leq \left|\int_0^p \left(\int_0^{\infty} y^{-1-a}\,((y))\cos(b\log xy)\,dy\right)\frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}\,dx\right|$$

$$\leq \int_0^p \left|\left(\int_0^{\infty} y^{-1-a}\,((y))\cos(b\log xy)\,dy\right)\right|\left|\frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}\right|dx$$

$$\leq \int_0^p \left(\int_0^{\infty} y^{-1-a}\,((y))\,dy\right)\left|\frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}\right|dx$$

Now we consider the integral with respect to y

(43) $\int_{0}^{\infty} y^{-1-a}$ ((y)) dy

$= (\lim t \to 0) \int_{0}^{1-t} y^{-1-a} \times y\, dy + (\lim t \to 0) \int_{1+t}^{\infty} y^{-1-a}$ ((y)) dy

(where t is a very small arbitrary positive number) .(Note that ((y))=y whenever $0 \le y < 1$).

Thus we have $(\lim t \to 0) \int_{1+t}^{\infty} y^{-1-a}$ ((y)) dy $< (\lim t \to 0) \int_{1+t}^{\infty} y^{-1-a}\, dy = \frac{1}{a}$

and $(\lim t \to 0) \int_{0}^{1-t} y^{-1-a} \times y\, dy = \frac{1}{1-a}$

Hence the integral (43) $\int_{0}^{\infty} y^{-1-a}$ ((y)) dy is bounded.

Since $\left| \int_{0}^{\infty} y^{-1-a} ((y)) \cos (b \log xy) dy \right| \le \int_{0}^{\infty} y^{-1-a}$ ((y)) dy , we conclude that the integral $\left| \int_{0}^{\infty} y^{-1-a} ((y)) \cos (b \log xy) dy \right|$ is a bounded function of x . Let this function be H(x) . Thus we have

(44) $\left| \int_{0}^{\infty} y^{-1-a} ((y)) \cos (b \log xy) dy \right| = H(x) \le K$ (K is a positive number)

Now equation (44) gives us

(45) $-K \le \int_{0}^{\infty} y^{-1-a} ((y)) \cos (b \log xy) dy \le K$

According to equation (42) we have

(46) $\iint\limits_{I3} F(x,y)\, dx\, dy = \int_{0}^{p} \left(\int_{0}^{\infty} y^{-1-a} ((y)) \cos (b \log xy)\, dy \right) \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^{a}} dx$

$\ge \int_{0}^{p} (-K) \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^{a}} dx = K \int_{p}^{0} \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^{a}} dx$

Since $\iint\limits_{I3}$ F(x,y) dx dy is bounded, then $\int_{p}^{0} \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^{a}} dx$ is also bounded. Therefore the integral

(47) G = $\int_0^p \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}$ dx is bounded

We denote the integrand of (47) by

(48) F = $\frac{1}{x^a}$ { (($\frac{1}{x}$))- x^{2a-1} }

Let δ G [F] be the variation of the integral G due to the variation of the integrand δ F. Since

(49) G [F] = $\int$ F dx (the integral (49) is indefinite)

(here we do not consider a as a parameter, rather we consider it as a given exponent)

We deduce that $\frac{\delta G[F]}{\delta F(x)} = 1$

that is

(50) δ G [F] = δ F (x)

But we have

(51) δ G[F] = $\int$ dx $\frac{\delta G[F]}{\delta F(x)}$ $\delta F(x)$ (the integral (51) is indefinite)

Using equation (50) we deduce that

(52) δ G[F] = $\int$ dx $\delta F(x)$ (the integral (52) is indefinite)

Since G[F] is bounded across the elementary interval [0,p] , we must have that

(53) δ G[F] is bounded across this interval

From (52) we conclude that

(54) $\delta G = \int_0^P$ dx $\delta F(x) = \int_0^P$ dx $\frac{dF}{dx}$ δx = [F δx] (at x = p)- [F δx] (at x = 0)

Since the value of [F δx] (at x = p)is bounded, we deduce from equation (54) that

(55) lim (x $\rightarrow 0$) F δ x must remain bounded.

Thus we must have that

(56) (lim x $\rightarrow 0$) [δ x $\frac{1}{x^a}$ { (($\frac{1}{x}$))- x^{2a-1} }] is bounded .

First we compute

(57) (lim x$\rightarrow 0$) $\frac{\delta x}{x^a}$

Applying L 'Hospital ' rule we get

(58) (lim x$\rightarrow 0$) $\frac{\delta x}{x^a}$ = (lim x$\rightarrow 0$)$\frac{1}{a}$x x^{1-a} $\times \frac{d(\delta x)}{dx} = 0$

We conclude from (56) that the product

(59) $0 \times (\lim x \to 0)$ { (($\frac{1}{x}$))- x^{2a-1} }must remain bounded.

Assume that a =0.5 .(remember that we considered a as a given exponent)This value a =0.5 will guarantee that the quantity { (($\frac{1}{x}$))- x^{2a-1} }

will remain bounded in the limit as $(x \to 0)$ .Therefore , in this case (a=0.5) (56) will approach zero as $(x \to 0)$ and hence remain bounded .

Now suppose that a< 0.5 .In this case we consider a as a parameter. Hence we have

(60) G_a [x]= $\int$ dx $\frac{F(x,a)}{x}$ x (the integral (60) is indefinite)

Thus

(61) $\frac{\delta G_a[x]}{\delta x} = \frac{F(x,a)}{x}$

But we have that

(62) $\delta G_a[x] = \int dx \frac{\delta G_a[x]}{\delta x} \delta x$ (the integral (62) is indefinite)

Substituting from (61) we get

(63) $\delta G_a[x] = \int dx \frac{F(x,a)}{x} \delta x$ (the integral (63) is indefinite)

We return to equation (49) and write

(64) G = lim (t $\to$ 0) $\int_t^p Fdx$ (t is a very small positive number 0<t<p)

= { F x(at p) – lim (t $\to$ 0) Fx (at t) } - lim (t $\to$ 0) $\int_t^p$ x dF

Let us compute

(65) lim (t $\to$ 0) Fx (at t) = lim (t $\to$ 0) t^{1-a} (($\frac{1}{t}$)) – t^a = 0

Thus equation (64) reduces to

(66) G – Fx (at p) = - lim (t $\to$ 0) $\int_t^p$ x dF

Note that the left – hand side of equation (66) is bounded. Equation (63) gives us

(67) δG_a =lim (t$\to$ 0) $\int_t^p$ dx $\frac{F}{x} \delta x$

(t is the same small positive number 0<t<p)

We can easily prove that the two integrals $\int_t^p$ x dF and $\int_t^p$ dx $\frac{F}{x}\delta x$ are absolutely convergent .Since the limits of integration do not involve any variable , we form the product of **(66)** and **(67)**

(68) $K = \lim(t \to 0) \int_t^p \int_t^p \; x dF \times dx \frac{F}{x}\delta x = \lim(t \to 0) \int_t^p \; F dF \times \int_t^p \; \delta x dx$

(K is a bounded quantity)

That is

(69) $K = \lim(t \to 0)\, [\, \frac{F^2}{2}(\text{ at p }) - \frac{F^2}{2}(\text{at t})\,] \times [\, \delta x \text{ (at p)} - \delta x \text{ (at t)}\,]$

We conclude from this equation that

(70) $\{\, [\frac{F^2}{2}(\text{ at p}) - \lim(t \to 0)\, \frac{F^2}{2}(\text{at t})\,] \times [\, \delta x \text{ (at p)}\,]\, \}$ is bounded .

(since $\lim(x \to 0)\, \delta x = 0$, which is the same thing as $\lim(t \to 0)\; \delta x = 0$)

Since $\frac{F^2}{2}$(at p) is bounded , we deduce at once that $\frac{F^2}{2}$ must remain bounded in the limit as $(t \to 0)$, which is the same thing as saying that F must remain bounded in the limit as $(x \to 0)$. Therefore .

(71) $\lim\,(x \to 0)\, \dfrac{((\frac{1}{x})) - x^{2a-1}}{x^a}$ must remain bounded

But

(72) $\lim\,(x \to 0)\, \dfrac{((\frac{1}{x})) - x^{2a-1}}{x^a} = \lim(x \to 0)\, \dfrac{x^{1-2a}}{x^{1-2a}} \times \dfrac{((\frac{1}{x})) - x^{2a-1}}{x^a}$

$$= \lim(x \to 0) \frac{x^{1-2a}((\frac{1}{x})) - 1}{x^{1-a}} = \lim(x \to 0)\, \frac{-1}{x^{1-a}}$$

It is evident that this last limit is unbounded. This contradicts our conclusion (71) that

$$\lim\,(x \to 0)\, \frac{((\frac{1}{x})) - x^{2a-1}}{x^a} \text{ must remain bounded (for a< 0.5)}$$

Therefore the case a<0.5 is rejected .We verify here that ,for a = 0.5 (71)remains bounded as $(x \to 0)$.

We have that

(73) $((\frac{1}{x})) - x^{2a-1} < 1 - x^{2a-1}$

Therefore

(74) $\lim(a \to 0.5)\,(x \to 0)\,\frac{((\frac{1}{x})) - x^{2a-1}}{x^{a}} < \lim(a \to 0.5)\,(x \to 0)\,\frac{1 - x^{2a-1}}{x^{a}}$

We consider the limit

(75) $\lim(a \to 0.5)\,(x \to 0)\,\frac{1 - x^{2a-1}}{x^{a}}$

We write

(76) a = ($\lim x \to 0$) (0.5 + x)

Hence we get

(77) $\lim(a \to 0.5)\,(x \to 0)\; x^{2a-1} = \lim\,(x \to 0)\; x^{2(0.5+x)-1} = \lim\,(x \to 0)\, x^{2x} = 1$

(Since $\lim(x \to 0)\; x^{x} = 1$)

Therefore we must apply L 'Hospital ' rule with respect to x in the limiting process (75)

(78) $\lim(a \to 0.5)\,(x \to 0)\,\frac{1 - x^{2a-1}}{x^{a}} = \lim(a \to 0.5)\,(x \to 0)\,\frac{-(2a-1)x^{2a-2}}{ax^{a-1}}$

$= \lim(a \to 0.5)\,(x \to 0)\,\frac{(\frac{1}{a} - 2)}{x^{1-a}}$

Now we write again

(79) a = ($\lim x \to 0$) (0.5 + x)

Thus the limit (78) becomes

(80) $\lim(a \to 0.5)\,(x \to 0)\,\frac{(\frac{1}{a} - 2)}{x^{1-a}} = \lim\;(x \to 0)\,\frac{(0.5+x)^{-1} - 2}{x^{0.5-x}} = \lim\;(x \to 0)\,\frac{(0.5+x)^{-1} - 2}{x^{0.5} \times x^{-x}}$

$= \lim\;(x \to 0)\,\frac{(0.5+x)^{-1} - 2}{x^{0.5}}$ (Since $\lim\,(x \to 0)\, x^{-x} = 1$)

We must apply L 'Hospital ' rule

(81) $\lim\;(x \to 0)\,\frac{(0.5+x)^{-1} - 2}{x^{0.5}} = \lim\;(x \to 0)\,\frac{-(0.5+x)^{-2}}{0.5x^{-0.5}} = \lim\;(x \to 0)\,\frac{-2 \times x^{0.5}}{(0.5+x\;)^{2}} = 0$

Thus we have verified here that ,for a = 0.5 (71) approaches zero as ($x \to 0$) and hence remains bounded.

We consider the case a >0.5 . This case is also rejected, since according to the functional equation, if ($\zeta(s)$=0) (s = a+ bi) has a root with a>0.5 ,then it must have another root with another value of a < 0.5 . But we have already rejected this last case with a<0.5

Thus we are left with the only possible value of a which is a = 0.5

Therefore a = 0.5

This proves the Riemann Hypothesis

Conclusion.

The Riemann Hypothesis **is now proved.**

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