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# Nonuniform Multiwavelet Packets associated with Nonuniform Multiresolution Analysis with Multiplicity D

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## Abstract

In this paper we construct nonuniform multiwavelet packets associated with the nonuniform multiresolution analysis (NUMRA) with multiplicity D based on the theory of one dimensional spectral pairs, which is a generalization of NUMRA introduced by Gabardo and Nashed. Further, we obtained an orthonormal basis for $L^2(\mathbb{R})$ from the collection of dilation and transilation of nonuniform multiwavelet packets as a generalization of nonuniform multiwavelet packets, that generalizes a result of Behera on wavelet packets associated with NUMRA.

**Keywords:** NUMRA with multiplicity D; nonuniform Multiwavelet; wavelet packets.

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## 1. Introduction

One of the most useful methods to construct a wavelet is through the concept of multiresolution analysis ($MRA$) introduced by Meyer and Mallat. The notion of MRA and wavelets were generalized to many different settings. One can replace the dilation factor 2 by any integer $N \geq 2$. In general, in higher dimensions, it can be replaced by a dilation matrix $A$, in this case the number of wavelets required is $|detA| - 1$. But in all these cases, the translation set is always a group. Recently, Gabardo and Nashed in [7] defined a multiresolution analysis associated with a translation set $\{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1$, $r$ is an odd integer and $r, N$ are relatively prime, a discrete set which is not necessarily a group. They call this an $NUMRA$. As, the case $N = 1$ reduces to the standard definition of $MRA$ with dyadic dilation. In [12] we obtain an NUMRA with multiplicity $D$, we called it $NUMRA - D$ that generalizes a particular case of a result of Calogero and Garrig´os [3] on biorthogonal $MRA$'s of multiplicity $D$ in nonstandard setup. A study with respect to $NUMRA$ has been done by many authors in the references [8-9, 11-15].

As in the case of classical wavelet bases, they have poor frequency localization. To overcome with this problem Meyer and Wickerhauser [5] constructed wavelet packets which provides better frequency localization for large $j$. The concept of wavelet packet was introduced by Coifman, Meyer and Wickerhauser [5, 6]. Let $\{V_j : j \in \mathbb{Z}\}$ be an $MRA$ of $L^2(\mathbb{R})$ with corresponding scaling function $\varphi$ and wavelet $\psi$. Let $W_j = span\{\psi_{jk} : k \in \mathbb{Z}\}$ be the corresponding wavelet subspaces. Since the space $V_1$ splits into two orthogonal components $V_0$ and $W_0$, we can also split $W_j$ into two orthogonal subspaces, which is span of $\{\psi(2^j . - k) : k \in \mathbb{Z}\} = \left\{\psi(2^j(. - \frac{k}{2^j})) : k \in \mathbb{Z}\right\}$, each of them can further be split into two parts. Repeating the splitting procedure $j$ times, $W_j$ is decomposed into $2^j$ subspaces each generated by integer translates of a single function. If we apply this to each $W_j$ , then we have a basis of $L^2(\mathbb{R})$, which will consist of integer translates of a countable number of functions. This basis is called the wavelet packet basis. Taking tensor product this construction was extended to the higher dimension by Coifman and Meyer [4]. The nontensor product version is due to Long and Shen [10]. Behra in [2] discuss the corresponding results in higher dimensions associated with a general matrix dilation and several scaling functions.

Adopting similar procedure Behera study wavelet packets associated with nonuniform multiresolution analysis in [1] and multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$ in [2]. Motivated from the work of Behera on wavelet packets associated with $NUMRA$, in this article we construct the multiwavelet packets associated with the nonuniform multiresolution analyses with multiplicity $D$. We also identify the sub collections of this system which form orthonormal bases for $L^2(\mathbb{R})$. Further we investigate their properties by means of the Fourier transform.

## 2. Basic definitions and notation

In this section, we will state some important preliminaries and notation that we are need to construct multiwavelet packets associated with nonuniform multiresolution analysis with multiplicity $D$ ($NUMRA-D$), where $D$ is a positive integer. As mentioned before $NUMRA-D$ is a generalization of $MRA$ as well as $NUMRA$.

By a nonuniform orthonormal multiwavelet $\Psi = \{\psi^l\}_{l=1}^{L}$ (*simply, nonuniform multiwavelet*) in $L^2(\mathbb{R})$ associated with the dilation $2N$ and the translation set $\Lambda_{r,N}$ , we mean that the family

$$\mathcal{A}(\Psi) \equiv \left\{(2N)^{j/2}\,\psi\left((2N)^j\ \cdot\ -\lambda\right):\ \psi \in \Psi, j \in \mathbb{Z}, \qquad \lambda \in \Lambda_{r,N}\right\}$$

forms a complete orthonormal system for $L^2(\mathbb{R})$. In the situation of $N = 1$, $r = 1$ and the nonuniform multiwavelet $\Psi$ is known as orthonormal multiwavelet associated with the dilation 2 and the translation set $\mathbb{Z}$ while $\Psi$ is known as orthonormal wavelet (*simply, wavelet*) in the case of $L = 1$. The generalization of the notion of $NUMRA$ into $NUMRA-D$ with respect to the dilation $\theta \equiv 2Na$ and translation $\Lambda_{r,N}$ as $2Na\,\Lambda_{r,N} \subset \Lambda_{r,N}$ , where a is defined as follows:

$$a = \begin{cases} \dfrac{n+1}{2} & :\ n \in \mathbb{N}\ when\ N\ =\ 1, \\ n & :\ n \in \mathbb{N}\ when\ N\ >\ 1, \end{cases}$$

provides more possibilities of dilation factors with respect to the translation set $\Lambda_{r,N}$ . $NUMRA-D$ in [12] for translation set $\Lambda \equiv \Lambda_{r,N}$ is defined as follows:

**Definition 2.1.** A nonuniform multiresolution analysis with multiplicity $D$ for dilation $\theta \equiv 2Na$ and translation $\Lambda$, is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following axioms:

$(P1)\ V_j\ \subset V_{j+1},\ for\ all\ j\ \in\ \mathbb{Z},$

$(P2) f(\cdot) \in V_j\ \ \ if\ and\ only\ if\ f(\theta\ \cdot) \in V_{j+1},\ for\ all\ j\ \in \mathbb{Z},$

$(P3)\ \bigcap_{j\ \in\mathbb{Z}} V_j\ =\ \{0\},$

$(P4)\ \bigcup_{j\ \in\mathbb{Z}} V_j\ \ is\ dense\ in\ L^2(\mathbb{R})$, and

(P5) There exist functions $\varphi^1, \varphi^2, \ldots, \varphi^D\ \in V_0$, called the scaling functions, such that the collection $\{\varphi^d(\cdot - \lambda):\ \lambda \in \Lambda, 1 \leq\ d \leq\ D\}$ is a complete orthonormal basis for $V_0$.

In the axiom $(P5)$, the set of scaling functions $\Phi \equiv \{\varphi^1, \varphi^2, \ldots, \varphi^D\}$ is called multiscaling function of multiplicity $D$ which generates an $NUMRA$ that leads to a nonuniform multiwavelet. A standard $NUMRA$ theory assumes that there is only one scaling function $\varphi$ whose $\lambda$-translates, $\lambda \in \Lambda$ constitutes a complete orthonormal basis of their span $V_0$.

For $D = 1$, $a = 1$ and $N \in \mathbb{N}$, the sequence $\{V_j\}_{j\in\mathbb{Z}}$ in the above definition is nothing but $NUMRA$ with integer dilation $2N$ and translation $\Lambda$ introduced by Gabardo and Nashed [7, 8] while for the case of $N = 1$ and $a \in \left\{\frac{n+1}{2}:\ n \in \mathbb{N}\right\}$, this is same as classical $MRA$ of multiplicity $D$ with dilation $2a$ and translation set $\mathbb{Z}$ (*simply call as* $MRA-D$). When $N \geq 1$, the dilation factor of $\theta$ ensures that $\theta\Lambda \subset 2\mathbb{Z} \subset \Lambda$.

In a similar fashion of the classical theory of multresolution analysis, another sequence $\{W_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is defined by $W_j = V_{j+1} \ominus V_j$ , $j \in \mathbb{Z}$ and $\ominus$ denotes the orthogonal complement of $V_j$ in $V_{j+1}$, for an $NUMRA-D$ $\{V_j\}_{j\in\mathbb{Z}}$ with dilation factor $\theta$. These subspaces inherit the scaling property of $\{V_j\}_{j\in\mathbb{Z}}$, and hence we have the following orthogonal decompositions:

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j = V_0 \oplus (\oplus_{j\geq 0} W_j) .$$

A set of functions $\{ \psi^l : 1 \leq l \leq (\theta - 1)D\} := \Psi$ in $L^2(\mathbb{R})$ is said to be a nonuniform multiwavelet associated with the $NUMRA-D$ $\{V_j\}_{j\in\mathbb{Z}}$ if the collection $\{ \psi^l(\cdot - \lambda) : 1 \leq l \leq (\theta - 1)D,\ \lambda \in \Lambda\}$ forms an orthonormal basis for $W_0$. We call $\Psi$ to be an $NUMRA-D$ multiwavelet. In view of properties of $W_j$ , the collection

$$\{\theta^{\frac{j}{2}} \psi^l(\theta^j \ \cdot -\lambda) : j \in \mathbb{Z}, 1 \leq l \leq (\theta - 1)D, \lambda \in \Lambda\}$$

forms an orthonormal basis for $L^2(\mathbb{R})$ if $\Psi$ is a nonuniform multiwavelet.

Below we mention a result obtained in [12] that will be used in sequel.

**Theorem 2.2 [12].** Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an $NUMRA-D$ with dilation $\theta$ and translation $\Lambda$. If there exist $L$ functions $\psi^k, 1 \leq k \leq L$, in $V_1$ such that the family of functions

$$V_1^{'} := \{\varphi^d(\cdot - \lambda), \psi^k(\cdot - \lambda): 1 \leq d \leq D;\ 1 \leq k \leq L;\ \lambda \in \Lambda\}$$

forms an orthonormal system for the generating subspace $V_1$, then, $L = (\theta - 1)D$ is a necessary and sufficient condition such that the above system is complete in $V_1$.

The following are some important findings that we will need for the present work.

**Proposition 2.3 [12].** Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an $NUMRA-D$ with dilation $\theta$ and translation $\Lambda$. Then the space $V_1$ consists precisely of the functions $f \in L^2(\mathbb{R})$ whose Fourier transform can be written as

$$\widehat{f}(\theta\xi) = \sum_{d=1}^{D} m^{f,d}(\xi)\, \widehat{\varphi^d}(\xi), \qquad a.e. \quad \xi \in \mathbb{R}, \tag{2.1}$$

for locally $L^2$ functions $m^{f,d}$ for $1 \leq d \leq D$ together with

$$m^{f,d}(\xi) = m_1^{f,d}(\xi) + e^{-\frac{2\pi i\xi r}{N}} m_2^{f,d}(\xi), \tag{2.2}$$

where $m_1^{f,d}$ and $m_2^{f,d}$ , $d = 1,2,\ldots,D$, are locally $L^2$, $1/2$-periodic functions.

In the above proposition the space $V_1$ consist functions $f \in L^2(\mathbb{R})$. Since $(1/\theta)f(x/\theta) \in V_0$, there exists a sequence $\{a_\lambda^f = (a_\lambda^{f,1}, a_\lambda^{f,2}, \ldots, a_\lambda^{f,D})\}_{\lambda\in\Lambda}$ satisfying $\sum_{d=1}^{D}\sum_{\lambda\in\Lambda}\left|a_\lambda^{f,d}\right|^2 < \infty$ such that

$$\frac{1}{\theta} f\left(\frac{x}{\theta}\right) = \sum_{d=1}^{D}\sum_{\lambda\in\Lambda} a_\lambda^{f,d}\, \varphi^d(x - \lambda), \tag{2.3}$$

or, equivalently, by taking the Fourier transform of both sides of the previous equation, we obtained the result (2.1) and (2.2), with

$$m^{f,d}(\xi) = \sum_{\lambda\in\Lambda} a_\lambda^{f,d} e^{-2\pi i\xi\lambda},$$

$$m_1^{f,d}(\xi) = \sum_{m\in\mathbb{Z}} a_{2m}^{f,d}\, e^{-4\pi i\xi m}, \ and \ m_2^{f,d}(\xi) = \sum_{m\in\mathbb{Z}} a_{2m+\frac{r}{N}}^{f,d}\, e^{-4\pi i\xi m}. \tag{2.4}$$

We denote by $\widehat{\Phi}$ and $\widehat{\Psi}$ column vectors in $C^D$ and in $C^L$ as $\widehat{\Phi} = \{\widehat{\varphi^1}, \ldots, \widehat{\varphi^D}\}$ and $\widehat{\Psi} = \{\widehat{\psi^1}, \ldots, \widehat{\psi^L}\}$, respectively. In particular, since $\varphi^d(x) \in V_0 \subset V_1$, from Proposition 2.3 there are locally $L^2$ functions $m_0^{\varphi^d,d'}$, for $1 \leq d, d' \leq D$, such that, for a.e.

$$\widehat{\varphi^d}(\theta\xi) = \sum_{d'=1}^{D} m_0^{\varphi^d,d'}(\xi)\,\varphi^{d'}(\xi)$$

$$= \sum_{d'=1}^{D}\left(m_{01}^{\varphi^d,d'}(\xi) + e^{-\frac{2\pi i\xi r}{N}}\, m_{02}^{\varphi^d,d'}(\xi)\right)\varphi^{d'}(\xi). \qquad (2.5)$$

Taking $m_{01}^{d,d'} \equiv m_{01}^{\varphi^d,d'}$ and $m_{02}^{d,d'} \equiv m_{02}^{\varphi^d,d'}$, in the matrix notation, this can be written as follows:

$$\widehat{\Phi}(\theta\xi) = M_0(\xi)\,\widehat{\Phi}(\xi), \ \ for\ a.e.\ \xi \in \mathbb{R},$$

with $M_0(\xi) = \left(M_{01}(\xi) + e^{-2\pi i\xi r/N} M_{02}(\xi)\right)$, where the matrices $M_{01}$ and $M_{02}$ defined by $M_{01} = \left(m_{01}^{d,d'}\right)_{1\le d,d'\le D}$ and $M_{02} = \left(m_{02}^{d,d'}\right)_{1\le d,d'\le D}$ are usually called low-pass filters (or scaling matrix filters) associated with the scaling family $\Phi$.

Similarly, in case of $\psi^l(x) \in W_0 \subset V_1$, where $W_0 = \overline{span}\{\psi^l(x-\lambda): 1 \le l \le L;\ \lambda \in \Lambda\}$, there are locally $L^2$ functions $m_1^{\psi^l,d}$, for $1 \le d \le D$ and $1 \le l \le L$ such that, for a.e. $\xi \in \mathbb{R}$,

$$\psi^l(\theta\xi) = \sum_{d=1}^{D} m_1^{\psi^l,d}(\xi)\widehat{\varphi^d}(\xi)$$

$$= \sum_{d=1}^{D}\left(m_{11}^{\psi^l,d}(\xi) + e^{-2\pi i\xi r/N}\, m_{12}^{\psi^l,d}(\xi)\right)\varphi^d(\xi), \qquad (2.6)$$

whose matrix notation is given as follows by considering $m_{11}^{l,d} \equiv m_{11}^{\psi^l,d}$ and $m_{12}^{l,d} \equiv m_{12}^{\psi^l,d}$ :

$$\widehat{\Psi}(\theta\xi) = M_1(\xi)\,\widehat{\Phi}(\xi), \ \ for\ a.e.\ \ \xi \in \mathbb{R},$$

with $M_1(\xi) = M_{11}(\xi) + e^{-2\pi i\xi r/N} M_{12}(\xi)$, where the matrices $M_{11}$ and $M_{12}$ defined by

$M_{11} = \left(m_{11}^{l,d}\right)_{1\le l\le L,1\le d\le D}$ and $M_{12} = \left(m_{12}^{l,d}\right)_{1\le l\le L,1\le d\le D}$

are usually called high-pass filters associated with the scaling family $(\Psi, \Phi)$.

**Proposition 2.4 [12].** Consider an $NUMRA-D$ with integer dilation $\theta$ and translation $\Lambda$ as in Definition 2.1. Then the following hold:

(i) The system $V := \{\varphi^d(\cdot - \lambda) : 1 \le d \le D;\ \lambda \in \Lambda\}$ which generates the space $V_0$ is orthonormal if and only if for a.e. $\xi \in \mathbb{R}$,

$$\sum_{p=0}^{\theta-1}\sum_{d=1}^{D}\left[m_{01}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{01}^{l,d}\left(\xi+\frac{p}{2\theta}\right)} + m_{02}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{02}^{l,d}\left(\xi+\frac{p}{2\theta}\right)}\right] = \delta_{kl}, \quad (2.7)$$

$$\sum_{p=0}^{\theta-1}\sum_{d=1}^{D}\alpha^P\left[m_{01}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{01}^{l,d}\left(\xi+\frac{p}{2\theta}\right)} + m_{02}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{02}^{l,d}\left(\xi+\frac{p}{2\theta}\right)}\right] = 0, \quad (2.8)$$

for $k, l = 1, \dots, D$, where $\alpha = e^{-\pi ir/N}$ and locally $L^2$, 1/2-periodic functions $m_{01}^{k,d}$, $m_{02}^{k,d}$, $k, d = 1, \dots, D$, are given by equations (2.4) and (2.5).

(ii) The system $W := \{\psi^k(\cdot - \lambda) : 1 \le k \le L;\ \lambda \in \Lambda\}$ which generates the space $W_0$ is orthonormal if and only if

$$\sum_{p=0}^{\theta-1}\sum_{d=1}^{D}\left[m_{11}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{11}^{l,d}\left(\xi+\frac{p}{2\theta}\right)} + m_{12}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{12}^{l,d}\left(\xi+\frac{p}{2\theta}\right)}\right] = \delta_{kl}, \quad (2.9)$$

$$\sum_{p=0}^{\theta-1}\sum_{d=1}^{D}\alpha^P\left[m_{11}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{11}^{l,d}\left(\xi+\frac{p}{2\theta}\right)} + m_{12}^{k,d}\left(\xi+\frac{p}{2\theta}\right)\overline{m_{12}^{l,d}\left(\xi+\frac{p}{2\theta}\right)}\right] = 0, \quad (2.10)$$

for a.e. $\xi \in \mathbb{R}$ and $k, l = 1, \ldots, L$, where locally $L2$, $1/2$-periodic functions $m_{11}^{k,d}, m_{12}^{k,d}$, $1 \leq k \leq L, 1 \leq d \leq D$ are given by equations (2.4) and (2.6).

**Proposition 2.5 [12].** Consider an $NUMRA - D$ by integer dilation $\theta$ as in Definition 2.1. Then the systems $V$ and $W$ generate the orthogonal subspaces $V_0$ and $W_0$, respectively if and only if for a.e. $\xi \in \mathbb{R}$ and $l = 1, \ldots, L,\ d = 1, \ldots, D,$

$$\sum_{h=1}^{D}\sum_{p=0}^{\theta-1} \overline{m_{11}^{l,h}\left(\xi + \frac{p}{2\theta}\right)}\, m_{01}^{d,h}\left(\xi + \frac{p}{2\theta}\right) + \overline{m_{12}^{l,h}\left(\xi + \frac{p}{2\theta}\right)}\; m_{02}^{d,h}\left(\xi + \frac{p}{2\theta}\right) = 0, \qquad (2.11)$$

and

$$\sum_{h=1}^{D}\sum_{p=0}^{\theta-1} \alpha^{P}\left[\overline{m_{11}^{l,h}\left(\xi + \frac{p}{2\theta}\right)}\, m_{01}^{d,h}\left(\xi + \frac{p}{2\theta}\right) + \overline{m_{12}^{l,h}\left(\xi + \frac{p}{2\theta}\right)}\; m_{02}^{d,h}\left(\xi + \frac{p}{2\theta}\right)\right] = 0, \qquad (2.12)$$

where locally $L^2$, $1/2$-periodic functions $m_{01}^{d,h}, m_{02}^{d,h}, m_{11}^{l,h}, m_{12}^{l,h}, 1 \leq d, h \leq D,\ \ 1 \leq l \leq L$, are given by equations (2.4), (2.5) and (2.6).

## 3. The splitting lemma for nonuniform multiwavelet

We constructed multiwavelet packets by continual splitting the wavelet subspace into finite number of orthogonal subspaces. This splitting is done with the help of the following lemma which is generalization of splitting lemma in [1], whose proof is easily obtained from proof of Theorem 3.2 in [7] and Propositions 2.3-2.5 given in [12]. In the splitting lemma we discuss the orthonormality of the system

$$V_1^{'} := \{\varphi^d(. - \lambda),\ \psi^k(\cdot - \lambda) :\ 1 \leq d \leq D;\ 1 \leq k \leq L;\ \lambda \in \Lambda\}.$$

For the sake of convenience we use single function $\psi_l^d$ to denote by $\varphi^d$ and $\psi^k$ by defining

$$\psi_l^d = \begin{cases} \varphi^d :\ for\ l = 0, 1 \leq d \leq D; \\ \psi^k : for\ l \neq 0;\ k = l + (d-1)(\theta - 1);\ 1 \leq d \leq D;\ 1 \leq l \leq (\theta - 1); \end{cases} \qquad (3.1)$$

also for $i \in \{1, 2\}$,

$$m_i^{d,l,h} = \begin{cases} m_{0i}^{d,h} : for\ l = 0, 1 \leq\ d, h\ \leq\ D; \\ m_{0i}^{k,h} : for\ l \neq 0;\ k = l + (d-1)(\theta - 1); 1 \leq d, h \leq D; 1 \leq l \leq (\theta - 1). \end{cases} \qquad (3.2)$$

**Lemma 3.1. (Splitting Lemma)** Let $\Phi = \{\varphi^1, \varphi^2, \ldots, \varphi^D\} \subset L^2(\mathbb{R})$, be a multiscaling function of multiplicity $D$ such that $\{\varphi^d(\cdot - \lambda): \lambda \in \Lambda,\ d = 1, \ldots, D\}$ is an orthonormal system in $L^2(\mathbb{R})$ and let $V = \overline{span}\{\theta^{l/2}\ \varphi^d(\cdot - \lambda) :\ \lambda \in \Lambda;\ d = 1, \cdots, D\}$. For $0 \leq l\ \leq (\theta - 1);\ 1 \leq d \leq D$; let

$$m^{d,l,h}(\xi) = \sum_{\lambda \in \Lambda} a_\lambda^{d,l,h}\ e^{-2\pi i\lambda\xi},\ \text{ with } \sum_{h=1}^{D}\sum_{\lambda \in \Lambda}\left|a_\lambda^{d,l,h}\right|^2 < 1$$

so that $m^{d,l,h}(\xi) = m_1^{d,l,h}(\xi) + e^{-2\pi i\xi r/N}\ m_2^{d,l,h}$.

Define

$$\widehat{\psi}_l^d(\xi) = \sum_{h=1}^{D} m^{d,l,h}\left(\frac{\xi}{\theta}\right)\widehat{\varphi}^h\left(\frac{\xi}{\theta}\right),\quad a.e.\ \xi \in \mathbb{R}.$$

Then $\left\{\psi_l^d\,(\cdot-\lambda)\,:\,1 \le d \le D;\ 0 \le l \le (\theta-1);\ \lambda \in \Lambda\right\}$ is an orthonormal system if and only if $m_1^{d,l,h}$ and $m_2^{d,l,h}$, $0 \le l \le (\theta-1)$; $1 \le d, h \le D$, satisfying equations (2.7) − (2.12) in the light of relations (3.2). Moreover this system is an orthonormal basis for $V$ if and only if it is orthonormal.

**Corollary 3.2.** Let $\left\{E_\lambda^d : 1 \le d \le D;\ \lambda \in \Lambda\right\}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$ and locally $L^2$, 1/2-Periodic functions $m_1^{d,l,h}$ and $m_2^{d,l,h}$, for $0 \le l \le (\theta-1)$; $1 \le d, h \le D$; be as defined in the splitting lemma, satisfying equations (2.7) − (2.12) in the light of relations (3.2). Define

$$F_\mu^{d,l} = \sum_{h=1}^{D}\sum_{\lambda\in\Lambda} \theta^{1/2}\, a_{\lambda-\theta\mu}^{d,l,h}\, E_\lambda^h, \mu \in \Lambda, 0 \le l \le (\theta-1);\ 1 \le d \le D.$$

Then $\left\{F_\mu^{d,l} : 0 \le l \le (\theta-1);\ 1 \le d \le D;\ \mu \in \Lambda\right\}$ is an orthonormal basis for its closed linear span $\mathcal{H}^{d,l}$ and $\mathcal{H} = \oplus_{l=0}^{\theta-1} \oplus_{d=1}^{D} \mathcal{H}^{d,l}$.

**Proof.** Let $\varphi^1, \varphi^2, \ldots, \varphi^D$, be function in $L^2(\mathbb{R})$ such that $\{\varphi^d(\cdot-\lambda) : \lambda \in \Lambda,\ d = 1, \ldots, D\}$ is an orthonormal system in $L^2(\mathbb{R})$. Let $V = \overline{span}\left\{\theta^{1/2}\,\varphi^d(\theta\cdot-\lambda) : \lambda \in \Lambda,\ d = 1, \ldots, D\right\}$. Define a linear operator $T$ from the Hilbert space $V$ to $\mathcal{H}$ by $T\left(\theta^{1/2}\,\varphi^h(\theta\cdot-\lambda)\right) = E_\lambda^h$ and let $\psi_l^d, 0 \le l \le (\theta-1);\ 1 \le d \le D$; be functions satisfying equation (2.3) ($f$ replaced by $\psi_l^d$ ).

Since $\psi_l^d(t) = \sum_{h=1}^{D}\sum_{\lambda\in\Lambda} \theta\, a_\lambda^{d,l,h}\, \varphi^h(\theta t-\lambda)$, we have

$$\psi_l^d(t-\mu) = \sum_{h=1}^{D}\sum_{\lambda\in\Lambda} \theta a_\lambda^{d,l,h}\, \varphi^h(\theta(t-\mu)-\lambda)$$

$$= \sum_{h=1}^{D}\sum \theta\, a_\lambda^{d,l,h}\, \varphi^h(\theta(t-\mu)-\lambda)$$

or, equivalently,

$$T\left(\psi_l^d(t-\mu)\right) = \sum_{h=1}^{D}\sum_{\lambda\in\Lambda} \theta^{1/2}\, a_{\lambda-\theta\mu}^{d,l,h}\, T\left(\theta^{1/2}\,\varphi^h(\theta\cdot-\lambda)\right)$$

$$= \sum_{h=1}^{D}\sum_{\lambda\in\Lambda} \theta^{1/2}\, a_{\lambda-\theta\mu}^{d,l,h}\, E_\lambda^h = F_\mu^{d,l}.$$

The result now established from splitting lemma. ■

## 4. Construction of nonuniform multiwavelet packets associated with NUMRA-D

Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an $NUMRA-D$ with dilation $\theta$ and translation $\Lambda$. As we observed, applying splitting lemma to the space

$$V_1 = \overline{span}\left\{\theta^{l/2}\,\varphi^d(\theta\cdot-\lambda) : \lambda \in \Lambda;\ d = 1,\cdots,D\right\},$$

we get the functions $\{\omega_l^d : 1 \le d \le D;\ 0 \le l \le (\theta-1)\}$, where

$\widehat{\omega}_l^d\,(\xi) = \sum_{h=1}^{D} m^{d,l,h}\left(\frac{\xi}{\theta}\right)\widehat{\varphi}^h(\xi/\theta), a.e.\ \ \xi \in \mathbb{R}.$ (4.1)

such that $\left\{\omega_l^d\,(\cdot-\lambda) : 0 \le l \le \theta,\ d = 1,..,D;\ \lambda \in \Lambda\right\}$, from the orthonormal basis for $V_1$. Now $\omega_0^d = \varphi^d$, $d = 1,..,D$ is a multiscaling function of multiplicity $D$ while $\omega_l^d$ , $1 \le l \le \theta-1$, $d = 1,\cdots,D$, are the basic multiwavelets associated with $NUMRA-D$. Now for any $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, we define $\omega_n^d : 1 \le d \le D$, recursively as follows. Suppose that $\{\omega_p^d : 1 \le d \le D,\ p \in \mathbb{N}_0\}$ are defined already. Then define $\omega_{q+\theta p}^d$ ,$0 \le q \le \theta - 1$, by

$$\omega_{q+\theta p}^d(x) = \sum_{h=1}^{D} \sum_{\lambda\in\Lambda} \theta\, a_\lambda^{d,q,h}\, \omega_p^h(\theta x - \lambda). \tag{4.2}$$

Note that (4.2) defines $\omega_n^d$ for $n \ge 0$. Taking Fourier transform we get

$$(\widehat{\omega}_{q+\theta p}^d)(\xi) = \sum_{h=1}^{D} m^{d,q,h}(\xi/\theta)\, \widehat{\omega}_p^h(\xi/\theta),\ 0 \le q \le \theta - 1,\ 1 \le d \le D. \tag{4.3}$$

By defining

$$F_p^d(\xi) = \left(\omega_p^1(\xi), \omega_p^2(\xi), \dots, \omega_p^D(\xi)\right)^t \text{ and}$$
$$\widehat{F}_p^d(\xi) = \left(\widehat{\omega}_p^1(\xi), \widehat{\omega}_p^2(\xi), \dots, \widehat{\omega}_p^D(\xi)\right)^t,$$

expression (4.3) can be written in matrix notation as

$$\widehat{F}_{q+\theta p}^d(\xi) = M_q(\xi/\theta)\ \widehat{F}_p^d\ (\xi/\theta). \tag{4.4}$$

where the square matrix $M_q(\xi)$ of order $D$ is given by

$$M_q(\xi) = \left(m^{d,q,h}(\xi)\right)_{1\le d\le D, 1\le h\le D}. \tag{4.5}$$

**Definition 4.1.** The function $\omega_l^d : 1 \le l \le L,\ 1 \le d \le D$ as defined above will be called the basic nonuniform multiwavelet packets corresponding to the $NUMRA-D$ $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ associated with the dilation $\theta$.

Note that (4.2) defines $\omega_l^d$ for every non-negative integer $l$ and every $d$ such that $1 \le d \le D$.

## 5. Fourier Transform of nonuniform multiwavelet packets associated with NUMRA-D

**Proposition 5.1.** Let $\{\omega_n^d : n \ge 0,\ 1 \le d \le D\}$ be the basic nonuniform multiwavelet packets and consider

$$n = \tau_1 + (\theta)\tau_2 + (\theta)^2\, \tau_3\ + \cdots + (\theta)^{j-1}\ \tau_j \tag{5.1}$$

where $0 \le \tau_i \le \theta - 1,\ 1 \le i \le j,\ \tau_j \ne 0$, be the unique expansion of the integer $n$ of length $j$ in base $\theta$. Then

$$\widehat{F}_n^d(\xi) = M_{\tau 1}\left(\frac{\xi}{\theta}\right) M_{\tau 2}(\xi/\theta^2) \cdots M_{\tau j}\left(\frac{\xi}{\theta^j}\right) \widehat{\Phi}^d(\xi/\theta^j). \tag{5.2}$$

**Proof.** We will prove it by induction on the length of $n$. Since $\omega_0^d = \varphi^d$ and $\omega_l^d, 1 \le l \le \theta - 1,\ 1 \le d \le D$, are the multiwavelet, it follows from (2.7) it is true for $n$ of length 1. Assume that the expression (5.2) holds for all integers $n$ of length $j$. Then an integer $m$ of length $j+1$ is of the form $m = \tau + \theta n$, where $0 \le \tau \le \theta - 1$, and $n$ has length $j$. Now from (4.3) and (5.2), we have

$$\widehat{F}_m^d(\xi) = \widehat{F}_{\tau+\theta n}^d(\xi)$$

$$= M_\tau(\xi/\theta)\ \hat{F}_n^d(\xi/\theta)$$
$$= M_\tau(\xi/\theta) M_{\tau_1}(\xi/\theta^2) M_{\tau_2}(\xi/\theta^3)\ \cdots M_{\tau_j}\left(\frac{\xi}{\theta^{j+1}}\right) \hat{\Phi}^d(\xi/\theta^{j+1}).$$

Since $m = \tau + \theta n = \tau + \tau_1\,\theta + \tau_2\,\theta^2 + \cdots + \tau_j\,\theta^j$, for $\hat{F}_m^d(\xi)$ we obtained desired expression and this completes the proof by induction. ■

**Theorem 5.2.** Let $\{\omega_n^d : n \geq 0,\ 1 \leq d \leq D\}$ be the basic nonuniform multiwavelet packets associated with the $A - D\ \{V_j\}$. Then

(i) $\{\omega_n^d(\cdot - \lambda) : (\theta)^j \leq n \leq (\theta)^{j+1} - 1,\ \lambda \in \Lambda,\ 1 \leq d \leq D\}$ is an orthonormal basis of $W_j$, $j \geq 0$.

(ii) $\{\omega_n^d(\cdot - \lambda) :\ 0 \leq n \leq (\theta)^j - 1,\ \lambda \in \Lambda, 1 \leq d \leq D\}$ is an orthonormal basis of $V_j$, $j \geq 0$.

(iii) $\{\omega_n^d(\cdot - \lambda) :\ n \geq 0,\ \lambda \in \Lambda,\ 1 \leq d \leq D\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

**Proof.** Here, the prove is given by induction on $j$. Since $\{\omega_n^d : 1 \leq n \leq \theta - 1,\ 1 \leq d \leq D\}$ are the basic multiwavelets, so (i) is true for $j = 0$. Assume that it holds for $j$. By (2.3) and the assumption, we have

$$\left\{\theta^{\frac{1}{2}} \omega_n^d(\theta \cdot - \lambda) : \lambda \in \Lambda,\ \theta^j \leq n \leq \theta^{j+1} - 1, 1 \leq d \leq D\right\}$$

is an orthonormal basis of $W_{j+1}$. Denote

$$E_n^d = \overline{span}\{\theta^{1/2} \omega_n^d(\theta \cdot\ - \lambda) :\ \lambda \in \Lambda, 1 \leq d \leq D\}$$

so that

$$\oplus_{d=1}^D F_{j+1}^d = \oplus_{d=1}^D \oplus_{n=(\theta)^j}^{(\theta)^{j+1}-1} E_n^d.$$

Applying the splitting lemma to $E_n^d$, we get functions $g_l^{n,d}$, $0 \leq l \leq \theta - 1$, defined by $\left(g_l^{d,n}\right)^\wedge(\xi) = \sum_{h=1}^D m^{d,l,h}(\xi/\theta)\,\hat{\omega}_n^d(\xi/\theta)$, $0 \leq l \leq \theta - 1$, such that $\{g_l^{d,n}(. - \lambda) :\ 0 \leq l \leq \theta - 1,\ \lambda \in \Lambda\}$ is an orthonormal basis of $E_n^d$. Let $n$ have the expansion as in (4.3). Then, using (5.2), we get

$$\hat{G}^d(\xi) = M_l(\xi \backslash \theta) M_{\tau_l}(\xi \backslash (\theta)^2) \cdots M_{\tau_j}\left(\xi \backslash (\theta)^{j+1}\right) \hat{\Phi}^d(\xi \backslash (\theta)^{j+1}).$$

Where $\hat{G}^d = \left(g_l^{d,n}\right)_{1 \leq l \leq \theta - 1, \theta^j \leq n \leq \theta^{j+1} - 1}$. But the expression on the right-hand side is precisely $\hat{\omega}_m^d(\xi)$, where $m = l + (\theta)\tau_l + (\theta)^2\tau_2 + \cdots + (\theta)^j\tau_j = l + \theta n$. Hence, we get $g_l^n = \omega_{l+\theta n}^d$. Since

$\{\omega_{l+\theta n}^d\ :\ 0 \leq l \leq \theta - 1,\ (\theta)^j \leq n \leq (\theta)^{j+1} - 1\} = \{\omega_n^d : (\theta)^{j+1} \leq n \leq (\theta)^{j+2} - 1, 1 \leq\ d \leq D\}$,

is orthonormal basis of $W_{j+1}$ we have proved (i) for $j + 1$ and the induction is complete. Further (ii) follows from the fact that $V_j = V_0 \oplus \cdots \oplus W_{j-1}$ and (iii) from the fact that $\overline{\cup V_j} = L^2(\mathbb{R})$. ■

## 6. General multiwavelet packets

**Definition 6.1.** Let $\{\omega_n^d : n \geq\ 0,\ 1 \leq d \leq D\}$ be the basic nonuniform multiwavelet packets associated with $NUMRA - D\ \{V_j\ :\ j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$. The collection of functions $\mathcal{N} = \{\theta^{j/2}\ \omega_n^d(\theta^j . - \lambda) :\ n\ \geq 0,\ j \in \mathbb{Z}$, $\lambda \in \Lambda$ is to be called the general nonuniform multiwavelet packets associated with NUMRA−D Vj, where $\mathcal{N}$ is overcomplete in $L^2(\mathbb{R})$.

At $j = 0,\ n \geq 0, \lambda \in \Lambda$ the subcollection of $\mathcal{N}$ gives the basic multiwavelet packet constructed above and with $n =\ 1, 2, \cdots,\ \theta -\ 1,\ j \in \mathbb{Z}, \lambda \in \Lambda$ the subcollection is a multiwavelet basis.

Now, we will characterize the subcollection of $\mathcal{N}$ which form orthonormal bases for $L^2(\mathbb{R})$. Begin with several decomposition of multiwavelet subspaces $W_j$ . For $n \geq 0,\ 1 \leq d \leq D$ and $j \in \mathbb{Z}$, define

$$U_j^{n,d} = \overline{span}\,\{\theta^{j/2}\,\omega_n^d(\theta^j\,.-\lambda) : \lambda \in \Lambda\}.$$

Note that

$$\oplus_{d=1}^D U_j^{0,d} = V_j \quad and \quad \oplus_{d=1}^D \oplus_{l=1}^{\theta-1} U_j^{l,d} = W_j\ ,$$

so

$$\oplus_{d=1}^D U_{j+1}^{n,d} = \oplus_{d=1}^D \oplus_{l=0}^{\theta-1}\ U_j^{l,d}$$

can generalize to decompose into $\theta$ orthogonal subspaces.

By the following result we can get different decompositions of wavelet subspaces $W_j\ ,\ j \geq 0$, and different bases of $L^2(\mathbb{R})$.

**Proposition 6.2.** For $n \geq 0$ and $j \in \mathbb{Z}$, we have

$$\oplus_{d=1}^D U_{j+1}^{n,d} = \oplus_{d=1}^D \oplus_{l=0}^{\theta-1} U_j^{l+\theta n,d}\ .$$

**Proof.** For $1 \leq d \leq D$; let $E_\lambda^d(x) = \theta^{\frac{j+1}{2}}\,\omega_n^d(\theta^{j+1}x - \lambda),\ \lambda \in \Lambda$. Then $\{E_\lambda^d : \lambda \in \Lambda\}$ is an orthonormal basis for the Hilbert space

$U_{j+1}^{n,d} = \overline{span}\,(\theta^{(j+1)/2})\,\omega_n^d\,(\theta^{j+1}.-\lambda)$. For $0 \leq l \leq \theta - 1$, define

$$F_\tau^{d,l}(x) = \sum_{h=1}^{L}\sum_{\lambda\in\Lambda}\theta^{1/2}\,a_{\lambda-\theta\tau}^{d,l,h}\,E_\lambda^h\ ,\ \ \tau \in \Lambda, 1 \leq d \leq D,$$

and

$$\oplus_{d=1}^D \mathcal{H}_l^d = \overline{span}\,\{F_\tau^{d,l} : \tau \in \Lambda, 1 \leq d \leq D\}.$$

Then, by Corollary 3.2, we have

$$\oplus_{d=1}^D U_{j+1}^{n,d} = \oplus_{d=1}^D \oplus_{l=1}^{\theta-1} \mathcal{H}_l^d\ .$$

Now

$$\begin{aligned} F_\tau^{d,l} &= \textstyle\sum_{h=1}^D \sum_{\lambda\in\Lambda}\theta^{1/2}\,a_{\lambda-\tau\theta}^{l,d,h}\,E_\lambda^h(x) \\ &= \textstyle\sum_{h=1}^D \sum_{\lambda\in\Lambda}\theta^{\frac{1}{2}}\ a_\lambda^{d,l,h}E_{\lambda+\theta\tau}^d(x) \\ &= \textstyle\sum_{h=1}^D \sum_{\lambda\in\Lambda}\theta^{(\theta+1)/2}\,a_\lambda^{d,l,h}\ \omega_n^d\big((\theta^{j+1}x-\lambda)-\theta\tau\big) \\ &= \theta^{\frac{j}{2}}\textstyle\sum_{h=1}^D \sum_{\lambda\in\Lambda}\theta^{\frac{1}{2}}a_\lambda^{d,l,h}\ \omega_n^d\big(\theta(\theta^j x-\tau)-\lambda\big), \\ &= \theta^{\frac{j}{2}}\,\omega_n^d\big(\theta^j x-\tau\big). \end{aligned}$$

by using (4.2). Therefore

$$\oplus_{d=1}^D \oplus_{l=0}^{\theta-1}\mathcal{H}_l^d = \oplus_{d=1}^D \oplus_{l=0}^{\theta-1} U_j^{\theta n+l,d}\ ,$$

and

$$\oplus_{d=1}^D U_{j+1}^{n,d} = \oplus_{d=1}^D \oplus_{l=0}^{\theta-1} U^{\theta n+l,d}. \qquad \blacksquare$$

By the following result we can construct many orthogonal bases of $L^2(\mathbb{R})$ of $NUMRA-D$.

**Theorem 6.3.** Let $j \geq 0$. Then, we have

$$W_j = \oplus_{d=1}^{D} \oplus_{l=1}^{\theta-1} U_j^{l,d}$$

$$W_j = \oplus_{d=1}^{D} \oplus_{l=\theta}^{\theta^2-1} U_{j-1}^{l,d}$$

$$\vdots$$

$$W_j = \oplus_{d=1}^{D} \oplus_{l=\theta^m}^{\theta^{m+1}-1} U_{j-m}^{l,d}, \quad m \leq j$$

$$\vdots$$

$$W_j = \oplus_{d=1}^{D} \oplus_{l=\theta^j}^{\theta^{j+1}-1} U_0^{l,d} .$$

**Proof.** By repeated application of the previous proposition we can get the proof. ■

**Theorem 6.4.** Let $\{\omega_n^d : n \geq 0, \ 1 \leq d \leq D\}$ be the basic nonuniform multiwavelet packets associated with a $NUMRA - D \{V_j : j \in \mathbb{Z}\}$ and $S \subset \mathbb{N}_0 \times \mathbb{Z}$. Then the collection $\mathcal{N}_s := \{\theta^{j/2} \omega_n^d(\theta^j . -\lambda) : \lambda \in \Lambda, (n,j) \in$ $S, 1 \leq d \leq D$

is an orthonormal basis of $L^2(\mathbb{R})$ if and only if $\{I_{n,j} : n,j \in S\}$ is a partition of $\mathbb{N}_0$, where $I_{n,j} = \{l \in$ $\mathbb{N}0$: $\theta j n \leq l \leq \theta j(n+1) - 1$.

**Proof.** By Proposition 6.2 and Theorem 3 in [1], we can establish the proof. ■

Note that from Theorm 5.2 (iii) we can write for a subsets $S$ of $\mathbb{N}_0 \times \mathbb{Z}$ that

$$\oplus_{d=1}^{D} \oplus_{(n,j)\in S} U_j^{n,d} = L^2(\mathbb{R}).$$

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