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On Weakly Berwald Finsler Special (α, β) −metric

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Abstract: In this paper, we study the special (α,β)-metric $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ on a manifold M. Then prove that F is of scalar flag curvature and isotropic S-Curvature if and only if it is isotropic Berwald metric with almost isotropic flag curvature.

Key Words: Isotropic Berwald curvature; S-curvature; Weak berwald metric; almost isotropic flag curvature.

1. Introduction

Curvatures are the central concept of Finsler geometry. For a Finsler manifold (M, F), the flag curvature is a function $K(P, y)$ to the tangent planes $P \subset T_x M$ and non zero $y \in P$. A Finsler metric F is of scalar flag curvature if for any non-zero vector $y \epsilon T_x M$, $K = K(x, y)$ is of independent P containing $y \epsilon T_x M$ (hence $K = \sigma(x)$ when F is Riemannian) It is of almost isotropic flag curvature if

$$K = \frac{3c_{x^m} y^m}{F} + \sigma\,, \tag{1.1}$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M. It is one of the important problems in Finsler geometry is to study and characterize Finsler manifolds of almost isotropic flag curvature [11].

To study the geometric properties of a Finsler metric, one also considers non-Riemannian quantities. In Finsler geometry, there are several important non-Riemannian quantities: the Cartan torsion C , the Berwald curvature B, the mean Landsberg curvature J and S-curvature S, etc ([6] [9] [13] [20]). these are geometric quantities which vanish for Riemannian metrics.

Among the non-Riemannian quantities, the S-curvature $S = S(x, y)$ is closely related to the flag curvature which constructed by Z. Shen for given comparison theorems on Finsler manifolds. An n-dimentional Finsler metric F is said to have isotropic S-curvature if

$$S = (n + 1)cF, \tag{1.2}$$

for some scalar function $c = c(x)$ on M. In [13], it is proved that if a Finsler metric F of scalar flag curvature is of isotropic S-curvature (1.2), then it has almost isotropic flag curvature (1.1).

The geodisc curves of a Finsler metric $F = F(x, y)$ on a smooth manifold M, are determined by $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \epsilon T_x M$ for any $x \epsilon M$.

A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i_{jkl} = c\{F_{y^jy^k}\delta^i_j + F_{y^ky^l}\delta^i_j + F_{y^ly^j}\delta^i_k + F_{y^jy^ky^ly^i}\}, \tag{1.3}$$

where $c = c(x)$ is a scalar function on M [6].

As a generalization of Berwald curvature, Basco-Matsumato proposed the notion of Douglas curvature [1]. A Finsler metric is called a Douglas metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x,y)y^i$. In order to find explicit examples of Douglas metrics, i.e we considered some (α,β) −metrics. An (α,β) −metric is a Finsler metric of the form $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$, where $\varphi = \varphi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a R iemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M.

In this paper, we consider special metric $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ with some non-Riemannian curvature properties and prove the following.

Theorem 1.1. Let $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ be a non-Riemannian special metric on a manifold M of dimention n. Then F is of scalar flag curvature with isotropic S-curvature (1.2), if and only if it has isotropic Berwald curvature (1.3) with almost isotropic flag curvature (1.1). In this case, F must be locally Minkowskian.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by TxM the tangent space at $x \in M$, by $TM = \cup_{x\epsilon M} TxM$ the tangent bundle of M and by $TM_0 = TM\setminus\{0\}$ the slit tangent bundle on M is a function $F : TM \to [0,\infty)$ which has the following properties:

a) $F\ is\ C^\infty on\ TM\setminus\{0\}$;

b) F is positively 1-homogeneous on the fibers of tangent bundle TM;

c) For each $y \in T_xM$, the following quadratic form g_y on T_xM

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y+su+tv)]|_s, t = 0, u, v\epsilon T_xM.$$

Let $x \in M$ and $F_x := F|\_T_xM$. To measure the non-Euclidean feature of F_x, define $C_y: T_xM \otimes T_xM \otimes T_xM \to R$ by

$$C_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[g_y + tw(u,v)\right]|_t = 0,\ u,v,w\in T_xM.$$

The family $C := \{C_y\}_{y\epsilon TM_0}$ is called the cartan torsion. It is well known that $C = 0$ if and only if F is Riemannian [17]. For $y\epsilon T_xM_0$, mean cartan torsion I_y by $I_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$, By Diecke theorem, F is Riemannian if and only if $I_y = 0$.

The horizantal covariant derivatives of I along geodiscs give rise to the mean Landsberg cur-vature $J_y(u) := J_i(y)u^i$, where $J_i := I_i|_s y^s$. A Finsler metric is said to be weakly Landsbergian if $J = 0$.

Given a Finsler manifold (M, F), then a global vector field G is induced by F on $T\,M_o$, which in a standard coordinate (x_i, y_i) for $T\,M_0$ is given by $G = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i := g^{il}\left[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k - \frac{\partial(F^2)}{\partial\partial x^l}\right], y\epsilon T_xM.$$

Let G is called the spray assosiated to (M, F). In local coordinates, a curve $c(t)$ is geodesic if and only if its coordinates $c^i(t)$ satisfy $\ddot{c}^i + 2G^i(\ddot{c}) = 0$,

For a tangent vector $y \in T_xM_0$, define $B_y: T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y: T_xM \otimes T_xM \to R$ by $B_y(u,v,w) := B^i_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x\ and\ E_y(u,v) := E_{jk}(y)u^jv^k, where$

$$B^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} = \frac{1}{2} B^m_{ijm}.$$

Let B and E are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $B = 0$ and $E = 0$, respectively.

A Finsler metric F is said to be isotropic mean Berwald metric if its mean Berwald curvature is in the following form

$$E_{ij} = \frac{n+1}{2F} c h_{ij},$$

where $c = c(x)$ is a scalar function on M and hij is the angular metric [6].

Define $D_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ by $D_y(u, v, w) \coloneqq D^i_{jkl}(y) u^i v^j w^k \frac{\partial}{\partial x^i}|_x$ where

$$D^i_{jkl} \coloneqq B^i_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk}, l y^i\}.$$

We call D:={ $D_y \coloneqq \{D_y\}_{y \in TM_0}$ the Douglas curvature. A Finsler metric with $D = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Basco-Matsumato as a generalization of Berwald metrics [1]. For a Finsler metric F on an n-dimentional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \dots . dx^n$ is defined by

$$\sigma F\ (x) = \frac{Vol(B^n(1))}{Vol\{(y^i) \varepsilon R^n | F(x, y) < 1\}}$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary func-tions, even F is locally expressed by elementary functions. Let G^i denote the geodisc coeffi-cients of F in the same local coordinate system. The S-curvature can be defined by

$$S(Y) \coloneqq \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}[In \sigma_F(x)],$$

where $Y = y^i \frac{\partial}{\partial x^i}|_x \epsilon T_x M.$. It is proved that $S = 0$ if F is a Berwald metric. There are many non-Berwald metrics satisfying $S = 0$. S said to be isotropic, if there is a scalar functions $c(x)$ on M such that $S = (n + 1)c(x)F$.

The Riemann curvature $R_y = R^i_k dx^k \otimes \partial x^i|_x : T_x M \to T_x M$ is a family of linear maps on tangent spaces, defined by

$$R^i_k = \frac{2\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^i \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

For a flag $P = span\{y, u\} \subset T_x M$ with flagpole y, the flag curvature $K = K(p, y)$ is defined by

$$K(P, y) \coloneqq \frac{g_y\left(u, R_y(u)\right)}{g_y(y, y) g_y(u, u) - g_y(y, u)^2}.$$

We say that a Finsler metric F is of scalar curvature if for any $y\ \varepsilon\ T_x M$, the flag curvature $K = K(x, y)$ is a scalar function on the slit tangent bundle $T\ M_0$. In this case , for some scalar function K on $T\ M_0$ the Riemann curvature is in the following form

$$R^i_k = K F^2\{\delta^i_k - F^{-1} F_{y^k} y^i\}.$$

If K=constant, then F is said to be of constant flag curvature. A Finsler metric F is called isotropic flag curvature, if $K = K(x)$.

3. Proof of theorem 1.1

Let $F = \alpha\, \varphi(s), s = \frac{\beta}{\alpha}$ be an $(\alpha, \beta) - metric,$ where $\varphi = \varphi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on a manifold M. Let

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \quad s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right),$$
$$s_j = b^i s_{ij}, \quad r_j = b^i r_{ij}.$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. Let

$$r_{i0} := r_{ij} y^j, \quad s_{i0} := s_{ij} y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

Put

$$Q = \frac{\varphi'}{\varphi - s\varphi'}, \qquad \Theta = \frac{(\varphi - s\varphi')\varphi' - s\varphi\varphi''}{2\varphi\left((\varphi - s\varphi') + (b^2 - s^2)\varphi''\right)}, \qquad \Psi = \frac{\varphi''}{2((\varphi - s\varphi') + (b^2 - s^2)\varphi'')}.$$

Then the S-curvature is given by

$$\boldsymbol{S} = \left[Q' - 2\Psi Q_s - 2(\Psi Q)'\left(b^2 - s^2\right) - 2(n+1)Q\Theta + 2\lambda\right] s_0$$
$$+2(\Psi + \lambda) r_0 + \alpha^{-1}\left[(b^2 - s^2)\Psi' + (n+1)\Theta\right] r_{00}. \qquad (3.2)$$

Let us put

$$\Delta = 1 + sQ + (b^2 - s^2)Q'$$
$$\Phi = -\{n\Delta + 1 + sQ\}(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'',$$

In [5], Cheng- Shen characterize (α, β) −metrics with isotropic S −curvature.

Lemma 3.1. ([5]) Let $F = \alpha\varphi(\frac{\beta}{\alpha})$ be an (α, β)-metric on an n-manifold.Then, F is of isotropic S-curvature $S = (n + 1)cF$, if and only if one of the following holds

(i) β satisfies

$$r_{ij} = \epsilon\{b^2 a_{ij} - b_i b_j\}, \; s_j = 0, \qquad (3.3)$$

Where $\epsilon = \epsilon(x)$ is a scalar function and $\varphi = \varphi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\varphi\Delta^2}{b^2 - s^2},$$

where k is a constant. In this case, $c = k\epsilon$.

(i) β satisfies

$$r_{ij} = 0, \; s_j = 0. \qquad (3.4)$$

In this case, $c = 0$.

Let

$$\Psi_1 = \sqrt{b^2 - s^2\, \Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}}{\Delta^{\frac{3}{2}}}\Phi\right]'},$$

$$\Psi_2 = 2(n + 1)(Q - sQ') + 3\frac{\Phi}{\Delta}.$$

$$\theta := \frac{Q - sQ'}{2\Delta}. \qquad (3.6)$$

Then the formula for the mean Cartan torsion of an (α, β) −metric is given by following

$$I_i = \frac{1}{2}\frac{\partial}{\partial y^i}\left[\frac{(n+1)\varphi'}{\varphi} - \frac{(n-2)s\varphi''}{\varphi - s\varphi'} - \frac{3s\varphi'' - (b^2-s^2)\varphi'''}{\varphi - s\varphi' + (b^2-s^2)\varphi''}\right]$$

$$I_i = -\frac{\Phi(\varphi - s\varphi')}{2\Delta\varphi\alpha^2}(\alpha b_i - sy_i). \tag{3.7}$$

In [7], it is proved that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α,β)-metrics. Hence, in the continue, we suppose that $\Phi \neq 0$.

Let $G^i = G^i(x,y)$ *and* $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x,y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we have

$$G^i = \bar{G}^i_\alpha + Py^i + Q^i,$$

where

$$P := \alpha^{-1}\Theta[-2Q\alpha s_0 + r_{00}]$$

$$Q^i := \alpha Q s^i_0 + \Psi[-2Q\alpha s_0 + r_{00}]b^i.$$

Simplifying (3.8) yields the following

$$G^i = \bar{G}^i_\alpha + \alpha Q s^i_0 + \Theta\{-2\alpha Q s_0 + r_{00}\}\left\{\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'}b^i\right\}, \tag{3.9}$$

Clearly, if β is parallel with respect to α $\left(r_{ij} = 0 \text{ and } s_{ij} = 0\right)$, then $P = 0$ and $Q^i = 0$. In this case, $G^i = \bar{G}^i_\alpha$ are quadratic in y, and F is a Berwald metric.

For an (α,β) −metric $F = \alpha\varphi(s)$, the mean Landsberg curvature is given by

$$J_i = -\frac{1}{2\Delta\alpha^4}\{\frac{2\alpha^2}{b^2 - s^2}\left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\right](s_0 + r_0)h_i$$
$$+\frac{\alpha}{b^2 - s^2}\left[\Psi_1 + s\frac{\Phi}{\Delta}\right](r_{00} - 2Q\alpha s_0)h_i + \alpha[-\alpha Q' s_0 h_i$$
$$+\alpha Q(\alpha^2 s_i - y_i s_0) + \alpha^2\Delta s_{i0}] + [\alpha^2(r_{i0} - 2\alpha Q s_i) - r_{00} - 2\alpha Q s_0)y_i]\frac{\Phi}{\Delta}\}. \tag{3.10}$$

Besides, they also obtained

$$\bar{J} = J_i b^i = -\frac{1}{2\Delta\alpha^2}\{\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0 + s_0)\}. \tag{3.11}$$

The horizontal covariant derivatives $J_{i;m}$ and $J_{i|m}$ of J_i with respect to F and α respectively are given by

$$J_{i;m} = \frac{\partial J_i}{\partial x^m} - J_l\Gamma^l_{im} - \frac{\partial J_i}{\partial y^l}N^l_m, J_{i|m} = \frac{\partial J_i}{\partial x^m} - J_l\bar{\Gamma}^l_{im} - \frac{\partial J_i}{\partial y^l}\bar{N}^l_m,$$

Where $\Gamma^i_{ij} = \frac{\partial G^l}{\partial y^i \partial y^j}, N^l_j = \frac{\partial G^l}{\partial y^j}$ *and* $\bar{\Gamma}^l_{ij} = \frac{\partial \bar{G}^i}{\partial y^i \partial y^j}, \bar{N}^l_j = \frac{\partial \bar{G}^l}{\partial y^j}$.

Then we have,

$$J_{i;m}y^m = \left\{J_{i|m} - J_l\left(\Gamma^l_{im} - \bar{\Gamma}^l_{im}\right) - \frac{\partial J_i}{\partial y^l}\left(N^l_m - \bar{N}^l_m\right)y^m\right\}$$
$$= J_{i|m}y^m - J_l\left(N^l_i - \bar{N}^l_i\right) - 2\frac{\partial J_i}{\partial y^l}(G^l - \bar{G}^l). \tag{3.12}$$

Let F be a Finsler metric of scalar flag curvature K. By Akbar-Zadeh's theorem it satisfies following

$$A_{ijk;s;m}y^s y^m + KF^2 A_{ijk} + \frac{F^2}{3}\left[h_{ij}K_k + h_{jk}K_j + h_{ki}K_j\right] = 0, \tag{3.13}$$

where $A_{ijk} = FC_{ijk}$ is the Cartan torsion and $K_i = \frac{\partial K}{\partial y^i}$ [2]. Contracting (3.13) with g^{ij} yields

$$J_{i;m}y^m + KF^2 I_i + \frac{n+1}{3}F^2 K_i = 0. \tag{3.14}$$

By (3.12) and (3.13), for an (α,β) −metric $F = \alpha\varphi(s)$ of constant flag curvature K , then

$$J_{i;m}y^m - J_l\frac{\partial(G^l-\bar{G}^l)}{\partial y^i} - \frac{2\partial J_i}{\partial y^l}(G^l-\bar{G}^l) + K\alpha^2\varphi^2 I_i = 0. \tag{3.15}$$

Contracting (3.15) with b^i implies that

$$\bar{J}_{|m} - J_i a^{ik} b_{k|m} y^m - J_l\frac{\partial(G^l-\bar{G}^l)}{\partial y^i}b^i - \frac{2\partial\bar{J}}{\partial y^l}(G^l-\bar{G}^l) + K\alpha^2\varphi^2 I_i b^i = 0. \tag{3.16}$$

There exists a relation between mean Berwald curvature E and the S-curvature S. Indeed, taking twice vertical covariant derivatives of the S-curvature gives rise the E-curvature. It is easy to see that, every Finsler metric of isotropic S-curvature (1.2) is of isotropic mean Berwald curvature (2.1). Now, is the equation $S = (n + 1cF$ equivalent to the equation $E=n+12cF-1h$?

Recently, Cheng-Shen prove that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature if and only if it is of isotropic E-curvature [4]. Then, Chun-Huan-Cheng [3] extend this equivalency to the Finsler metric $F = \alpha^{-m}(\alpha + \beta)^m + 1$ for every real constant m, including Randers metric .

To prove Theorem 1.1, we need the following.

Theorem 3.2. Let $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ be a special metric on a manifold M of dimention n. Then the following are equivalent

(i) F is of isotropic S-curvature, $S = (n + 1)c(x)F$;

(ii) F is of isotropic mean Berwald curvature, $E = \frac{n+1}{2}cF^{-1}h$;

; where $c = c(x)$ is a scalar function on the manifold M.

In this case, $S = 0$. Then β is a Killing 1-form with constant length with respect to α, that is, $r_{00} = 0$.

Proof: (i) → (ii) is obvious. Conversely, suppose that F has isotropic mean Berwald curvature, $E = \frac{n+1}{2}cF^{-1}h$. Then we have

$$S = (n + 1)[c(x)F + \eta], \tag{3.17}$$

where $\eta = \eta_i(x)y^i$ is a 1-form on M. For the special metric

$$Q = \frac{s^2+1}{s(s-2)}\ ,\ \Theta = -\frac{1}{2}\frac{s(s^3+3s-4)}{(s+s^2-1)(-s^3+2b^2)},\ \ \Psi = \frac{1}{-s^3+2b^2}. \tag{3.18}$$

By substituting (3.17) and (3.18) in (3.2), we have

$$S = [-\frac{2(-3s^4+2sb^2+2s^3-2b^2+2s^2b^2-2s^2)}{s^2(s-2)^2(-s^3+2b^2)} + \frac{2(-3s^6+4s^5+4s^2b^2-5s^4+8s^3+4sb^2-4b^2)}{s^2(-s^3+2b^2)^2(s-2)^2}\times$$
$$(b^2-s^2) - \frac{(n+1)(s^2+1)(-4+s^3+3s)}{(s-2)(-1+s+s^2)(-s^3+2b^2)} + 2\lambda]s_0 + 2\left[\frac{1}{-s^3+2b^2}+\lambda\right]r_0$$
$$-\left[\frac{3s^2(b^2-s^2)}{\alpha(-s^3+2b^2)^2}\right]r_{00} - \left[\frac{(n+1)s(s^3+3s-4)}{2\alpha(-1+s+s^2)(-s^3+2b^2)}\right]r_{00}.$$
$$(n+1)\left[c\alpha\left(1+s+\frac{1}{s}\right)+\eta\right]. \tag{3.19}$$

Multiplying (3.19) with $s(1+s+s^2)(s^3+2b^2)^2(s+2)\alpha^5$ implies that

$$M_1 + M_2\alpha^2 + M_3\alpha^4 + M_4\alpha^4 + M_5\alpha^8 + M_6\alpha^{10}\alpha[M_7 + M_8\alpha^2 + M_9\alpha^4 + M_{10}\alpha^6 + M_{11}\alpha^8 + M_{12}\alpha^{10}] = 0, \tag{3.20}$$

where

$$M_1 = \left[-\beta^2c(n+1) + 2\beta\lambda(s_0+r_0) - \beta\eta(n+1) + \frac{r_{00}}{2}(n+1)\right]\beta^9,$$

$M_2 = -\frac{1}{2}[10\beta^2 c(n+1) - 12\beta\lambda(s_0 + r_0) + 12\beta s_0 + 6\beta\eta(n+1) + 3r_{00}(n+3)]\beta^7,$

$M_3 = -[-5\beta^2 c(n+1) + 2\beta b^2 s_0(n+2) - 4\beta b^2\eta(n+1) + 8\beta\lambda b^2(s_0 + r_0) + 2\beta(s_0(2n+3) + r_0)$
$\quad +r_{00}(2n-1)(b^2+2)\beta^5,$

$M_4 = -2[-2\beta^2 b^4 c(n+1) - 2\beta b^4\eta(n+1) + 4\beta b^4\lambda(s_0 + r_0) - \beta\big((-ns_0 + 2r_0) + 3s_0\big) + 4\beta b^4\lambda(s_0 + r_0)$
$\quad +4\beta b^2\eta(n+1) - 8\beta b^2\lambda(s_0 + r_0) + 2\beta b^2\big((2n+3)s_0 + r_0\big) + r_{00}b^2(5n+8)]\beta^3,$

$M_5 = -2b^2[-4\beta c(n+1) + 10\beta b^2 c(n+1) - 12b^2\lambda(s_0 + r_0) + 3(ns_0 - 2r_0) + 6b^2\eta(n+1)]\beta^2,$

$M_6 = 20b^4 c(n+1)\beta,$

$M_7 = [2\beta\lambda(s_0 + r_0) + \beta s_0(n+1) - \beta\eta(n+1) + r_{00}(n+4)]\beta^8,$

$M_8 = [-4\beta^2 b^2 c(n+1) - 4\beta\lambda(s_0 + r_0) - 2\beta\eta(n+1) + 8\beta b^2\lambda(s_0 + r_0) + 2\beta(r_0 - 2n(\eta b^2 + s_0))$
$\quad +r_{00}n((b^2+5) - 2r_{00}(b^2+2))]\,\beta^6,$

$M_9 = [20\beta^2 b^2 c(n+1) - 2\beta^2 c(n+1) + 12\beta\eta b^2(n+1) - 24\beta b^2\lambda(s_0 + r_0) + 3\beta(ns_0 + 2r_0)$
$\quad 3\beta s_0(2b^2 - 3) + 3r_{00}b^2(4+n)]\beta^4,$

$M_{10} = 2b^2[-10\beta^2 c(n+1) - 2\beta\eta b^2(n+1) + 4\beta b^2\lambda(s_0 + r_0) + \beta(4s_0 n + 2r_0 + 9s_0) + 4r_{00}(n+1)]\beta^2,$

$M_{11} = 8b^2[b^2\eta(n+1) - 2b^2\lambda(s_0 + r_0) - r_0 + ns_0]\beta,$

$M_{12} = -8b^4 c(n+1).$

The term of (3.20) which is seemingly does not contain α^2 is M_1. Since β^9 is not divisible by α^2, then $c = 0$ which implies that

$$M_1 = M_7 = 0.$$

Therefore (3.20) reduces to following

$$M_2 + M_3\alpha^2 + M_4\alpha^4 + M_5\alpha^6 + M_6\alpha^8 = 0, \tag{3.21}$$

$$M_8 + M_9\alpha^2 + M_{10}\alpha^4 + M_{11}\alpha^6 + M_{12}\alpha^8 = 0. \tag{3.22}$$

By plugging $c = 0$ in M_2 and M_8, the only equations that don't contain α^2 are the following

$$-\beta[2\lambda(s_0 + r_0) - (n+1)\eta + 3r_{00}(n+3)] = \tau_1\alpha^2, \tag{3.23}$$

$$4\beta b^2[2\lambda(s_0 + r_0) - (n+1)\eta] + r_{00}(2n-1)(b^2+2) = \tau_2\alpha^2, \tag{3.24}$$

where $\tau_1 = \tau_1\alpha^2$ and $\tau_2 = \tau_2\alpha^2$ are scalar functions on M. By eliminating $[2\lambda(s_0 + r_0) - (n+1)\eta]$, we get

$$r_{00} = \tau\alpha^2, \tag{3.25}$$

where $\tau = \frac{\tau_2 - 4b^2\tau_1}{(b^2+2)(4b^2(2n-1)) - 3(n+3)}$.

By (3.23) or (3.24), it follows that

$$2\lambda(s_0 + r_0) - (n+1)\eta = 0. \tag{3.26}$$

By (3.25), we have $r_0 = \tau\,\beta$. Putting (3.25) and (3.26) in M_8 and M_9 yields

$$M_8 = [n(b^2+5) - 2(b^2+2)]\tau\alpha^2\beta^6, \tag{3.27}$$

$$M_9 = \big[[(6b^2 + 3n - 9)s_0 - 6r_0]\beta - 3b^2(n+4)r_{00}\tau\alpha^2\big]\beta^4. \tag{3.28}$$

By putting (3.27) and (3.28) into (3.22), we have

$$[(6b^2 + 3n - 9)s_0 - 6r_0]\beta^5 - 3b^2(n+4)r_{00}\tau\alpha^2\beta^4 + n(b^2+5) - 2(b^2+2)\tau\alpha^2\beta^6$$

$$-M_{10}\alpha^2 + M_{11}\alpha^4 + M_{12}\alpha^6 = 0. \tag{3.29}$$

The only equations of (3.29) that do not contain α^2 is $[n(b^2+5) - 2(b^2+2)\tau\beta + (6b^2+3n-9)s_0 - 6r_0]\beta^5$. Since β^6 is not divisible by α^2, then we have

$$[n(b^2+5) - 2(b^2+2)\tau\beta^6 + (6b^2+3n-9)s_0 - 6r_0] = 0. \tag{3.30}$$

By lemma 3.1, we always have $s_j = 0$. Then (3.30), reduces to following

$$[n(b^2+5) - 2(b^2+2)]\tau\beta - 6r_0 = 0. \tag{3.31}$$

Thus

$$[n(b^2+5) - 2(b^2+2)]\tau b_i - 6\tau b_i = 0. \tag{3.32}$$

By multiplying (3.32) with b^i, we have

$$\tau = 0.$$

Thus by (3.28), we get $\eta = 0$ and then $S = (n+1)cF$. By (3.25), we get $r_{ij} = 0$. Therefore lemma 3.1, implies that $S = 0$. This completes the proof.

Proof of Theorem 1.1: Let F be an isotropic Berwald metric (1.3) with almost isotropic flag curvature (1.1). In [22], it is proved that every isotropic Berwald metric (1.3) has isotropic S-curvature (1.2).

Conversely, suppose that F is of isotropic S-curvature (1.2) with scalar flag curvature K . In [13], it is showed that every Finsler metric of isotropic S-curvature (1.2) has almost isotropic flag curvature (1.1). Now, we are going to prove that F is a isotropic Berwald metric. In [6], it is proved that F is an isotropic Berwald metric (1.3) if and only if it is a Douglas metric with isotropic mean Berwald curvature (2.1). On the other hand, every Finsler metric of isotropic S-curvature (1.2) has isotropic mean Berwald curvature (2.1). Thus for completing the proof, we must show that F is a Douglas metric. By proposition 3.2, we have $S = 0$. Therefore by theorem 1.1 in [13], F must be of isotropic flag curvature $K = \sigma(x)$. By proposition 3.2, β is a Killing 1-form with constant length with respect to α, that is, $r_{ij} = s_j = 0$. Then (3.9), (3.10) and (3.11) reduce to

$$G^i - \bar{G}^i = \alpha Q s_0^i, \qquad J_i = -\frac{\Phi s_{i0}}{2\alpha\Delta}, \quad \bar{J} = 0.$$

By (3.8), we get

$$I_i b^i := -\frac{\Phi(\varphi - s\varphi')}{2\Delta F}(b^2 - s^2).$$

Now we consider two cases:

Case I: $dimM \geq 3$. In this case, by Schur lemma F has constant flag curvature and (3.6) holds, the equation (3.16) reduces to following

$$\frac{\Phi s_{i0}}{2\Delta\alpha} a^{ik} s_{k0} + \frac{\Phi s_{l0}}{2\Delta\alpha}\left(sQs_0^l + Q' s_0^l (b^2 - s^2)\right) - KF\frac{\Phi}{2\Delta}(\varphi - s\varphi')(b^2 - s^2) = 0. \tag{3.33}$$

By assumption $\Phi \neq 0$. Thus by (3.32), we get

$$s_{i0}s_0^i + s_{l0}\left(\alpha Q s_0^l\right)_{.i} b^i - KF\alpha\,(\varphi - s\varphi')(b^2 - s^2) = 0. \tag{3.34}$$

The following holds

$$(\alpha Q s_0^l)_i b^i = sQs_0^i + Q' s_0^i (b^2 - s^2) = 0.$$

Then (3.34) can be rewritten as follows

$$s_{i0}s_0^i\Delta - K\alpha^2\,\varphi(\varphi - s\varphi')(b^2 - s^2) = 0. \tag{3.35}$$

By (3.6), (3.18) and (3.35), we obtain

$$\left[1 + \frac{s^2+1}{s-2} - \frac{2(b^2-s^2)(-1+s+s^2)}{s^2(s-2)^2}\right] s_{i0}s_0^i - K\alpha^2\left[\frac{(-1+s+s^2)(s-2)}{s^2}(b^2 - s^2)\right] = 0. \tag{3.36}$$

Multiplying (3.36) with $-s^2(s-2)^2\alpha^5$ yields

$$A+\alpha B=0,$$

where

$$A=-K20b^2\beta\alpha^6+\left(5K\beta^3b^2+2b^2\beta s_{i0}s_0^i+20K\beta^3\right)\alpha^4+\left(\beta^3s_{i0}s_0^i-5K\beta^5+K\beta^5b^2\right)\alpha^2$$
$$-K\beta^7-s_{i0}s_0^i\beta^5$$

$$B=8Kb^2\alpha^6+\left(10Kb^2\beta^2-8K\beta^2-2s_{i0}s_0^ib^2\right)\alpha^4+\left(-5Kb^2\beta^4+2s_{i0}s_0^ib^2\beta^2-10K\beta^4\right)\alpha^2$$
$$+\left(5K\beta^6-s_{i0}s_0^i\beta^4\right).$$

Obviously, we have $A=0$ and $B=0$.

If $A=0$ and the fact that β^7 is not divisible by α^2, we get $K=0$. Therefore (3.36) reduces to following

$$s_{i0}s_0^i=a_{ij}s_0^js_0^i=0.$$

Because of positive-definiteness of the Riemannian metric α, we have $s_{i0}=0$, i.e., β is closed. By $r_{00}=0$ and $s_0=0$, it follows that β is parallel with respect to α. Then $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ is a Berwald metric. Hence F must be locally Minkowskian.

Case II: Let $dim\, M=2$. Suppose that F has isotropic Berwald curvature (1.3). In [6], it is proved that every isotropic Berwald metric [3] has isotropic S-curvature, $S=(n+1)cF$.

By proposition 3.2, $c=0$. Then by [3], F reduces to a Berwald metric. Since F is non-Riemannian, then by Szabo's rigidity theorem for Berwald surface (see [2] page 278), F must be locally Minkowskian.

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