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Weights for commutators of oscillatory type integral operators

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Abstract

This paper devote to set up some weighted estimates for commutators(with symbol depending on the related weights)formed by one kind of oscillatory integral operators and weighted BMO functions.

Keywords: Ap weight; oscillatory integral; commutator; BMO.

1. Introduction

In many mathematical fields, such as Ergodic Theory, PDE, one-sided operators are required. For the weighted boundedness of one-sided operators, one can refer to [1, 5, 6, 10, 11, 13, 14]. As an essential part of singular integral in harmonic analysis, oscillatory integral has many kinds of versions in its appearance, such as the Fourier transform, the Bochner-Riesz means, the Radon transform in CT technology and so on. Besides the closely relationship with classical harmonic analysis, the applications to number theory and PDE were another two impetus to study the oscillatory singular integrals. The study of weights for one-sided operators was motivated not only as the generalization of the theory of both-sided ones but also their natural appearance in harmonic analysis, for example, it is required when we treat the one-sided Hardy-Littlewood maximal operator

$$M^{+}f(x)=\sup_{h>0}\int_{x}^{x+h}|f(y)|dy;\ M^{-}f(x)=\sup_{h>0}\int_{x-h}^{x}|f(y)|dy$$

arising in the ergodic maximal function. The good one-sided weights for M^{+} and M^{-} was first introduced by Sawyer [13]

$$A_p^{+}:\begin{cases} M^{-}w\le Cw,\ p=1 \\ \sup_{a<b<c}\dfrac{1}{(c-a)^p}\displaystyle\int_a^b w(x)dx\left(\int_b^c w(x)^{1-p'}dx\right)^{p-1}<\infty,1<p<\infty. \end{cases}$$

and

$$A_p^{-}:\begin{cases} M^{+}w\le Cw,\ p=1 \\ \sup_{a<b<c}\dfrac{1}{(c-a)^p}\displaystyle\int_b^c w(x)dx\left(\int_a^b w(x)^{1-p'}dx\right)^{p-1}<\infty,1<p<\infty. \end{cases}$$

It is interesting to note that

$$A_p^{+}\cap A_p^{-}=A_p,A_p\subset A_p^{+},A_p\subset A_p^{-}.$$

For more definitions and results, we refer to [10, 11, 13]. Here and after, A_p denotes the Muckenhoupt classes. Without loss of generality, we only consider the right-hand-side operator in the following discuss.

We emphasize that the one-sided weight classes A_p^+ not only control the boundedness of M^+, but also they are the right weight classes for one-sided Calder´on-Zygmund singular integrals,

$$\tilde{T}^+ f(x) = \lim_{\varepsilon\to 0^+} \int_{x+\varepsilon}^{\infty} K(x-y)f(y)dy \ ,$$

where K is a Calder´on-Zygmund kernel with support in $R^- = (-\infty, 0)$ (cf. [1]).

Based on the definition of the above one-sided operators, Fu, Lu and Shi first gave the definition of one-sided oscillatory singular integral operator T^+ in [2]. We recall the definition of one-sided oscillatory integral operator as

$$T^+ f(x) = \lim_{\varepsilon\to 0^+} \int_{x+\varepsilon}^{\infty} e^{iP(x,y)}K(x-y)f(y)dy$$

and

$$T^- f(x) = \lim_{\varepsilon\to 0^+} \int_{\infty}^{x-\varepsilon} e^{iP(x,y)}K(x-y)f(y)dy \,,$$

where $P(x,y)$ is a real-valued polynomial defined on $R\times R$ and the kernel K is a one-sided Calderon- Zygmund kernel with support in R^- and R^+, respectively. Let $b \in L^1(R)$ and $w \in A^\infty$. We say $b \in BMO_w$ if

$$\|b\|_{BMO_w} = \sup_I \frac{1}{w(I)} \int_I |b - b_I| < \infty,$$

where I denote any bounded interval and

$$b_I = \frac{1}{|I|}\int_I b.$$

Besides BMO_w function, Lipschitz function $Lip_\beta, 0<\beta<1$ can also characterize the symbol of one-sided operator's commutators

$$\|b\|_{Lip_\beta(R)} = \sup_{h\neq 0} \frac{|b(x+h) - b(x)|}{|h|^\beta} < \infty..$$

Here is another way of stating $b \in Lip_\beta$. For any $x, y \in R$, we say $b \in Lip_\beta$, if

$$|b(x) - b(y)| \le \|b\|_{Lip_\beta} |x-y|^\beta \ .$$

For $b \in Lip_\beta$, the commutators formed by M^+ and b were defined by [6]

$$\left(M^+_b f\right)(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |b(x) - b(y)|\,|f(y)|dy.$$

An observation of the definitions of M^+_b, we can define the commutators formed by T^+ and $b \in BMO_w$ as

$$T_b^+ f(x) = \lim_{\varepsilon\to 0^+} \int_{x+\varepsilon}^{\infty} e^{iP(x,y)}K(x-y)\,(b(x) - b(y))f(y)dy \,.$$

Now, we may formulate our results as follows

Theorem 1.1. Let $1 < p < \infty, u \in A_p, v \in A_p^+, w = (\frac{u}{v})^{1/p}$ and $b \in BMO_w$. Then there exists a constant $C > 0$ such that

$$\left\| T_b^+ f \right\|_{L^p(v)} \leq C \left\| f \right\|_{L^p(u)}$$

holds for all bounded f with compact support.

Following Theorem 1.1, we give a (L^p, L^p) boundedness of T_b^+ when $b \in Lip_\beta$.

Theorem 1.2. Let $1 < p < \infty, w \in A_p^+, w(.)(.)^{\beta p} \in A_p$ and $b \in Lip_\beta$. Then there exists a constant $C > 0$ such that

$$\left\| T_b^+ f \right\|_{L^p(v)} \leq C \left\| f \right\|_{L^p(u)}$$

holds for all bounded f with compact support.

Remark 1.3. Theorem 1.1 and Theorem 1.2 are absolutely different from that in [14], which allows us to give results for commutators of higher order. On the other hand, we can not obtain the results in [14] from Theorem 1.1 and Theorem 1.2 since we can not take $u = v$. Since the oscillatory factor $e^{iP(x,y)}$ makes it impossible to established the norm inequalities of oscillatory singular integral by the methods as in the case of Calder´on-Zygmund operators, the proofs of our theorems are different from that of in [7] and [8] which dealt with the one-sided maximal functions and one-sided Calder´on-Zygmund singular integral operators.

Section 2 contains some lemmas which are essential to our proofs. In section 3, we set up the proof of Theorem 1.1 and Theorem 1.2. Throughout this paper, the letter C will denote a positive constant which may vary from line to line but will remain independent of the relevant quantities.

2. Some lemmas

Lemma 2.1 ([8]). Let $1 < p < \infty, u \in A_p, v \in A_p^+, w = (\frac{u}{v})^{1/p}$ and $b \in BMO_w$. Then there exists a constant $C > 0$ such that the inequality

$$\left\| M_b^+ f \right\|_{L^p(v)} \leq C \left\| f \right\|_{L^p(u)}$$

holds for all bounded f with compact support.

Lemma 2.2 ([4]). Let $0 < \beta < 1$. Then there exists the following relationship between the space BMO_w and Lip_β: $BMO_w = Lip_\beta$ as $w = t^\beta$.

Lemma 2.3 ([3]). Let $w \in A_p^+$. Then there exists $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_p^+$.

To prove Theorem 1.1, the interpolation theorem of operators with change measures plays an important role, which formed by Stein and Weiss in [15].

Lemma 2.4. Suppose that u_0, v_0, u_1, v_1 are positive weight functions and $1 < p_0, p_1 < \infty$.. Assume sublinear operator S satisfies:

$$\left\| Sf \right\|_{L^{p_0}(u_0)} \leq C_0 \left\| f \right\|_{L^{p_0}(v_0)}, \left\| Sf \right\|_{L^{p_1}(u_1)} \leq C_1 \left\| f \right\|_{L^{p_1}(v_1)}.$$

Then

$$\|Sf\|_{L^p(u)} \le C\|f\|_{L^p(v)}$$

holds for any $0 < \theta < 1, \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, , where

$$u = u_0^{\frac{p\theta}{p_0}} u_1^{\frac{p(1-\theta)}{p_1}} \; v = v_0^{\frac{p\theta}{p_0}} v_1^{\frac{p(1-\theta)}{p_1}}, C \le C_0^{\theta} C_1^{1-\theta}.$$

3. Proofs of main results

Observing that the proof of Theorem 1.2 is the combination of Theorem 1.1 and Lemma 2.3, we only need to show Theorem 1.1. We shall carry out the argument by induction. First, we assume the conclusion of Theorem 1.1 is valid for all polynomials which are the sums of monomials of degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y .Thus $P(x,y)$ can be written as

$$P(x,y) = a_{kl}x^k y^l + R(x,y) or P(x,y) = x^k Q_k(y) + R_0(x,y),$$

where $R(x,y) = \sum_{\alpha<k,\beta\le l} a_{\alpha\beta}x^{\alpha}y^{\beta} + \sum_{\beta<l} a_{k\beta}x^k y^{\beta}$ satisfying the above induction assumption and $Q_k(y) = \sum_{|\beta|<l} a_{k\beta}y^{\beta}$ is polynomial in y of degree less than l, $R_0(x,y)$ has x-degree less than k. We split the kernel K as

$$K(x) = K(x)\chi_{\{|x|\le 1\}}(y) + K(x)\chi_{\{|x|>1\}}(y) := K_0 + K_\infty$$

and consider the corresponding splitting

$$T_b^+ f(x) = p.v. \int_x^\infty K_0(x-y)(b(x)-b(y))f(y)dy$$

$$+\int_x^\infty K_\infty(x-y)(b(x)-b(y))$$

$$:= T_{b,0}^{\ \ +} + T_{b,\infty}^{\ \ +}.$$

Take any $h \in R$, and write $P(x,y) = a_{kl}(x-h)^k(y-h)^l + R(x,y,h)$, where the polynomial $R(x,y,h)$ satisfies the induction assumption, and the coefficients of $R(x,y,h)$ depend on h.

3.1. Estimates for $T_{b,0}^{\ \ +}$.

It is an easy observation that

$$T_{b,0}^{\ \ +} f(x) := T_{b,01}^{\ \ +} f(x) + T_{b,02}^{\ \ +} f(x).$$

Now we split f into three parts as follows

$$f(y) = f(y)\chi_{\{|y-h|<1/2\}}(y) + f(y)\chi_{\{1/2\le|y-h|<5/4\}}(y) + f(y)\chi_{\{|y-h|\ge 5/4\}}(y)$$

$$= f_1 + f_2 + f_3$$

When $|x-h| < 1/4$, , one has

$$\int_{|x-h|<1/4} |T_{b,01} f_1(x)|^p v(x)dx \le C \int_{|x-h|<1/2} |f(y)|^p u(y)dy,$$

where C is independent of h. The fact that

$$\left|T_{b,01}^{+}f_2(x)\right| \le CM_b^{+}(f_2)(x).$$

Lemma 2.1 shows that

$$\int_{|x-h|<1/4} \left|T_{b,01}f_2(x)\right|^p v(x)dx \le C \int_{|x-h|<5/4} |f(y)|^p u(y)dy,$$

where C is independent of h. We notice that $T_{b,01}f_3(x) = 0$. Therefore

$$\int_{|x-h|<1/4} \left|T_{b,01}f(x)\right|^p v(x)dx \le C \int_{|x-h|<5/4} |f(y)|^p u(y)dy,$$

where C is independent of h. On the other hand,

$$\int_{|x-h|<1/4} \left|T_{b,02}f(x)\right|^p v(x)dx \le C \int_{|x-h|<5/4} |f(y)|^p u(y)dy,$$

by the similar analysis. We thus have obtained

$$\int_{|x-h|<1/4} \left|T_{b,0}f(x)\right|^p v(x)dx \le C \int_{|x-h|<5/4} |f(y)|^p u(y)dy$$

holds uniformly, which implies

$$\left\|T_{b,0}^{+}f\right\|_{L^p(v)} \le C\left\|f\right\|_{L^p(u)}.$$

3.2. Estimates for $T_{b,\infty}^{+}$.

The analysis for the bound of $T_{b,\infty}^{+}$ is almost the same as that of [3], which follows from the method using in [9] and [12]. Our result differ from the previous one only in that we set up it based on one-sided singular integrals and the weights. For $j \ge 1$, we have

$$\left|T_{b,j}^{+}f(x)\right| \le CM_b^{+}(f)(x),$$

where C is independent of j. By lemma 2.1 and lemma 2.4, we know that there exists $\varepsilon > 0$, such that $v^{1+\varepsilon} \in A_p^{+}, u^{1+\varepsilon} \in A_p^{+}$. Thus we have

$$\left\|T_{b,j}^{+}f\right\|_{L^p(v^{1+\varepsilon})} \le C\left\|f\right\|_{L^p(u^{1+\varepsilon})},$$

where C is independent of j. On the other hand, by means of the methods in [3], we get

$$\left\|T_{b,j}^{+}f\right\|_{L^p(R)} \le C2^{-j\delta}\left\|f\right\|_{L^p(R)},$$

where C is dependents only on the total degree of $P(x,y)$, and $\delta > 0$. By Lemma 2.5, it follows that

$$\left\|T_{b,j}^{+}f\right\|_{L^p(v)} \le C2^{-j\theta\delta}\left\|f\right\|_{L^p(u)},$$

where $0 < \theta < 1$, θ is independent of j, and C depends only on the total degree of $P(x,y)$. Now we have porved

$$\left\| T_b^{+} f \right\|_{L^p(v)} \leq C \left\| f \right\|_{L^p(u)} ,$$

where C is dependents only on the total degree of $P(x, y)$.

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