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Supplement-Duo Modules

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Abstract

In this note we consider a generalization of the notion of duo modules namely supplement duo modules. Where an R module M is called a supplement duo module if every supplement submodule of M is a fully invariant. Many results about this concept are given. Also we study when the direct sum of supplement duo modules is supplement duo, and relationships between a supplement dou module and other modules.

Keywords: Duo modules; supplement duo modules; weak duo modules; supplement submodules.

Introduction

Throughout this paper all rings are commutative rings have an identity and all modules are untital. R is a ring and M a left R-module. A submodule N of an R-module M is called small denoted by $N \ll M$, if whenever $N + L = M$ for some submodule L of M, then $L = M$ .A submodule N of M is called fully invariant if $f(N) \subseteq N$ for every R-endomorphism f of M. Clearly 0 and M are fully invariant submodules of M. The R-module M is called a duo module if every submodule of M is a fully invariant. The ring R is a duo ring if it is duo as R-module. Asubmodule N is called a supplement of L if $N + L = M$ and, $N \cap L \ll N$, M is called supplemented, if every submodule of M has supplement in M.

An R-module M is called a weak duo module provided every direct summand submodule of M is fully invariant [1].

In this paper we introduce supplement duo module, where an R- module M is called a supplement duo if every supplement submodule of M is fully invariant

It is well-known that every direct summand of a module M is supplement then every supplement duo module is weak duo. In this paper we study supplement duo module, and we give some conditions under which supplement duo and weak duo are equivalent.

We need the following lemmas

Lemma 1 [2]: Let M be an R-module and N be a supplement submodule in M. Then every supplement submodule in N is a supplement submodule in M.

Lemma 2 [3]: Let M be an R-module and let N and A be submodules of M such that $N \subseteq A$. If N is supplement in M then N is supplement in A.

Lemma 3: Let M be an R-module, and let $N \subseteq A$ be submodules of M such that if N is a supplement in M and $\frac{A}{N}$ is supplement in $\frac{M}{N}$, then A is supplement in M.

Lemma 4: [4] If A and B are supplement submodules of K and L in R-modules M and N respectively, then $A \oplus B$ is supplement of $K \oplus L$ in $M \oplus N$.

Definition (1):

An R-module M is a supplement duo module if every supplement submodule of M is a fully invariant.

Remarks and Examples (2):

1-It is clear that every duo module is supplement duo and every supplement duo is weakly duo, but the converse are not true in generals, for example:
The Z-module Q is not duo, for the submodule Z of Q is not fully invariant,
But Q has only two direct summands namely (0) and Q. Hence Q is a supplement duo and a weak- duo Z-module.

2- The direct sum of supplement duo modules may not be supplement duo
The Z-module $Z_2 \oplus Z_2$ is not weak-duo, since there exists f: $Z_2 \oplus Z_2 \longrightarrow Z_2 \oplus Z_2$ defined by $f(\bar{x}, \bar{y}) = (\bar{y}, \bar{x}),\ \forall\, \bar{x}, \bar{y} \in Z_2$. So if $N = Z_2 \oplus (\bar{0})$ (which is a direct summand of $Z_2 \oplus Z_2$), then $f(N) = (\bar{0}) \oplus Z_2 \not\subseteq N$. Also, $Z_2 \oplus Z_2$ is not a supplement duo module.

Proposition (3):

A direct summand of supplement duo module is a supplement duo module.

Proof:

Let N be a direct summand of a supplement duo R-module M. Then $M = N \oplus W$ for some $W \leq M$. Let K be a supplement submodule of N and let $f: N \longrightarrow N$ be an R- homomorphism module. Since N is a direct summand, then N is a supplement submodule in M, hence K is a supplement submodule in M[Lemma 1].

Define h: $M \longrightarrow M$ by h(x)=f(x) for every x in N and zero otherwise

h is a well-defined R-homomorphism. It follows that $h(K) \subseteq K$, since M is a supplement duo module and K is a supplement submodule in M. But h(K) = f(K). Hence $f(K) \subseteq K$; that is K is a fully invariant submodule of N. Thus N is a supplement duo module.

An R-module is called quasi-projective if for each R-module N and every epimorphism $g: M \longrightarrow M$ and every R-homomorphism $f: M \longrightarrow N$, there is $f' \in End(M)$ such that $g \circ f' = f$, [5]

Proposition (4): Let M be a supplement duo R-module. If M is a quasi-projective then for any supplement submodule K of M, $\frac{M}{K}$ is a supplement duo R-module.

Proof:

Let $\frac{H}{K}$ be a supplement submmodule in $\frac{M}{K}$ and let h: $\frac{H}{K} \to \frac{M}{K}$ be an R-homomorphism.

Let π:M $\to \frac{M}{K}$ be a natural epimorphism .Since $\frac{M}{K}$ is quasi-projective then there exists g:M$\to$ M such that πog =hoπ.Then g(m)+K= h(m+K) for every m in M . $\frac{H}{K}$ supplement submmodule in $\frac{M}{K}$ and K supplement submodule of M, then by lemma 3 H is a supplement in M .So g(H)$\subseteq$H ,but M is supplement duo module ,thus h($\frac{H}{K}$)=g(H)+K.

Then $\frac{H}{K}$ is supplement duo module.

Proposition (5):

Let M be an R-module such that every cyclic submodule is supplement. Then M is a supplement duo module if and only if for each f $\in$ End(M) and for each m $\in$ M, there exists r $\in$ R such that f(m) = rm.

Proof:

$\Rightarrow$) Let f $\in$ End(M), m $\in$ M. Since <m> is supplement, then f(<m>) $\subseteq$ <m>. Then we are done.

$\Leftarrow$) The stated condition implies f(N) $\subseteq$ N for every f $\in$ End(M). It follows that M is a duo module. Hence it is a supplement duo module.

Proposition (6):

Let a module M = $M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2 such that M is a supplement duo module. Then Hom(M_1,M_2) = 0.

Proof:

Since M_1 is a direct summand of M, then M_1 is a supplement submodule in M. But M is a supplement duo module, so M_1 is a fully invariant submodule in M. Hence Hom(M_1,M_2) = 0 [1,lemma 1.9].

Theorem (7):

Let M = $M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2 such that annM_1 + annM_2 = R. Then M is a supplement duo module if and only if M_1 and M_2 are supplement duo modules and Hom(M_i,M_j) = 0 for i $\neq$ j, i, j $\in$ {1,2}.

Proof:

$\Rightarrow$) It follows by proposition 3 and proposition 6.

$\Leftarrow$) Let N be a supplement submodule of M. Since annM_1 + annM_2 =R, then by [6,Theorem 4.2N = $N_1 \oplus N_2$ for some $N_1 \leq M_1$, $N_2 \leq M_2$. It is esey to show , N_1 is a supplement submodule in M_1 and N_2 is a supplement submodule in M_2.by lemma(1,2,3) Let h:M $\longrightarrow$ M be an R-homomorphism. $\rho_j \circ h \circ i_j$:N_J $\longrightarrow$ N_J , j = 1,2, where ρ_j is the canonical projection and i_j is the inclusion map.Since M_1 , M_2 are supplement duo modules Hence $\rho_j \circ h \circ i_j(N_j) \subseteq N_j$, j = 1,2, and since Hom(M_i,M_j) = 0 for i $\neq$ j, i, j $\in$ {1,2}.

Then h(N)=h(N_1)+h(N_2) $\subseteq \rho_j \circ h \circ i_j(N_1) + \rho_j \circ h \circ i_j(N_2) \subseteq N_1+N_2$=N.

Lemma (8):

Let M be an R-module such that $M = \oplus M_i$, $M_i \leq M$. If N is a fully invariant submodule of M, then $N = \oplus(N \cap M_i)$.

Theorem (9):

Let a module $M = \oplus M_i$ of submodules $M_i(i \in I)$. Then M is a supplement duo module if and only if

(1) M_i is a supplement duo module for all $i \in I$.

(2) $Hom(M_i,M_j) = 0$ for all $i \neq j$, $i, j \in I$.

(3) $N = \oplus(N \cap M_i)$ for each supplement submodule N of M.

Proof:

$\Rightarrow$) It follows by proposition 3, proposition 6 and lemma 8.

$\Leftarrow$) Let N be a supplement submodule of M. By (3),$N = \oplus(N \cap M_i)$ then $N \cap M_j$ is supplement submodule of in M_i. Let $f:M \longrightarrow M$. For any $j \in I$. Let i_j: $M_j \longrightarrow M$, $f:M \longrightarrow M$ and $\rho_j : M_j \longrightarrow M$ where i_j is the inclusion and ρ_j is the canonical projection.

Hence $\rho_j \circ f \circ i_j$: $M_j \longrightarrow M_j$ and so $\rho_j \circ f \circ i_j$ $(N \cap M_j) \subseteq N \cap M_j$ for each $j \in I$. By condition (2),$Hom(M_i,M_j) = 0$ $\forall$ $i \neq j$.

Hence $f(\oplus(N \cap M_j)) \subseteq \oplus(\rho_j \circ f \circ i_j)(N \cap M_j) \subseteq \oplus(N \cap M_j)$.

Then M_i is a supplement duo module for all $i \in I$.

Proposition (10):

Let M be a supplement duo module. If K is a direct summand of M and L is a supplement submodule of M, then $K \cap L$ is a supplement submodule of M.

Proof:

Since K is a direct summand of M, $M = K \oplus H$ for some $H \leq M$. Since M is a supplement duo module and L is a pure submodule, then L is a fully invariant. Hence $L = (L \cap K) \oplus (L \cap H)$. Thus $L \cap K$ is a direct summand of L, so $L \cap K$ is a supplement submodule in L. But L is a supplement submodule in M, hence $L \cap K$ is a supplement submodule in M.

The following remark is clear.

Remark (11):

Let M be a semisimple R-module. Then the following statements are equivalent:

(1) M is a duo-module.

(2) M is a supplement duo module,

(3) M is a weak duo module.

Recall that an R-module M is called a fully stable module if every submodule N of M is a stable submodule of M, where N is called a stable submodule if $f(N) \subseteq N$ for any R-homomorphism $f:N \longrightarrow M$,[6].

Note that every stable submodule is a fully invariant submodule. Hence we get the following result.

Corollary (12):

Let M be a semisimple R-module. Then the following statements are equivalent:

(1) M is a supplement duo module,

(2) M is a duo-module.

(3) M is a weak-duo module.

(4) M is a fully stable module.

Recall that an R-module M is said to be extending (or CS), if every submodule of M is essential in a direct summand of M, [7].

Or equivalently, An R-module M is extending if and only if every closed submodule is a direct summand, [8].

Recall that an R-module M is called a supplement extending module if every submodule of M is essential in a supplement submodule of M [9].

It is clear every extending module is a supplement extending module.

A submodule N of an R-module M is called closed if it has no proper essential extension.

Proposition (13):

Let M be a supplement extending R-module. Then the following statements are equivalent:

(1) M is a supplement duo

(2) Every closed submodule of M is a fully invariant.

(3) M is a weak duo module.

Proof:

(1) $\Rightarrow$ (2) Let N be a closed submodule of M. Since M is a supplement extending module, N is essential in a supplement submodule (say H). Hence N = H, that is N is a supplement submodule. It follows that N is a fully invariant, sinc M is a supplement duo module.

(2) $\Rightarrow$ (3) Let N be a direct summand of M. Hence N is a closed submodule of M [10] and so that N is a fully invariant submodule of M.

Corollary (14):

Let M be an extending module over a PIR. Then the following statements are equivalent:

(1) M is a supplement duo module,

(2) Every closed submodule of M is a fully invariant submodule of M.

(3) M is a weak-duo module.

Proof:

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) It follows by proposition 15.

(3) $\Rightarrow$ (1) Let N be a supplement submodule of M. Since R is a P.I.R then by [11,excersice15,p.242], N is closed, and since M is extending N is a direct summand and so by(3) N is a fully invariant submodule of M. Thus M is a supplement duo module.

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