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Value Distribution of q-Difference Polynomials

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Abstract

In this paper, we deal with the zero distribution of q-difference polynomials $\boldsymbol{P(f)\Delta_q^k f - a(z)}$ and $[\boldsymbol{P(f)\,\Delta_q^k f}]^{(m)} - \boldsymbol{a(z)}$, where *P(f)* is a nonzero polynomial of degree *l*, $\boldsymbol{q} \in \mathbb{C}\setminus\{\mathbf{0},\mathbf{1}\}$ is a constant, $\boldsymbol{l,m} \in \mathbb{N}_+$ and *a(z)* is a small function of *f*.

Keywords: Zero distribution; zero order; q-difference polynomial.

1. Introduction And Main Results

In this paper, it is assumed that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory. For example, the proximity function *m(r, f)*, counting function *N(r, f)*, characteristic function *T(r, f)*, the first and second main theorem etc.,(see [6,18,19]). In particular, we use *S(r, f)* to denote any quantity satisfying the condition: *S(r, f)=o(T(r, f))* for all *r* outside a possible exceptional set *E* of the finite logarithmic measure

$$\lim_{r\to\infty}\int_{(1,r]\cap E}\frac{1}{t}dt < \infty.$$

We also use $S_1(r,\ f)$ to denote any quantity that satisfies $S_1(r,\ f) = o(T(r,f))$ for all *r* on a set *F* of logarithmic density 1; the logarithmic density of a set *F* is defined by

$$\limsup_{r\to\infty}\frac{\int_{(1,r]\cap F}\frac{1}{t}dt}{\log r},$$

and the lower logarithmic density of a set *F* is defined by

$$\liminf_{r\to\infty}\frac{\int_{(1,r]\cap F}\frac{1}{t}dt}{\log r}.$$

A meromorphic function *a(z)* is called a small function with respect to *f(z)*, if *T(r, a)=S(r, f)*. The order of a meromorphic *f* is defined as

$$\rho(f) = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

The q-difference operator for a meromorphic function *f* is defined as

$$\Delta_q f(z) = f(qz) - f(z)(q \neq 0,1),\ \ \Delta_q^{k+1} f(z) = \Delta_q\left(\Delta_q^k f(z)\right), k \in N.$$

During the last decades, many mathematicians and mathematical researchers were denoted to studying the value distribution of meromorphic solutions of the non-autonomous Schroder q-difference equation

$$f(qz) = R(z, f(z)),$$

where the right-hand side is rational in both arguments(see[4,8,916]). But in recent years, they are interested in establishing difference and q-difference operator analogs of Nevanlinna theory in the complex plane $\mathbb{C}$. By using it, a number of papers studied the value distribution of difference and q-difference polynomials.

For a transcendental meromorphic function *f*, Hayman[7] first proposed the conjecture that if $n \geq 1$, then $f(z)^n f'(z)$ takes every finite nonzero value $a \in \mathbb{C}$ infinitely often. This conjecture has been successively solved by Hayman, Mues, Bergweiler and Eremenko(see[6,15,1]). In 2010, Zhang and Korhonen [21] studied the value distribution of q-difference polynomial of meromorphic (resp.entire) functions and obtained: if

$n \geq 6(\text{resp. } n \geq 2)$, then $f(z)^n f(qz)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often and q is a nonzero zero complex constant.

Afterwards, Zhang et al.[22] proved the following result.

Theorem A. Let *f(z)* be a transcendental meromorphic function of finite order. Suppose that *n*, *k* are positive integers and *c* is a non-zero complex number such that $\Delta_c^k f(z) \not\equiv 0$, $a(z) \not\equiv (0, \infty)$ is a small function with respect to *f(z)*. If $n \geq 3k + 5$, then $f(z)^n \Delta_c^k f(z) - a(z)$ has infinitely many zeros.

In [17], Xu-Liu-Cao considered the zero distribution of q-difference polynomials $P(f)f(qz+\eta)$ and $P(f)[f(qz+\eta) - f(z)]$, and established the following result.

Theorem B. Let *f(z)* be a zero-order transcendental meromorphic(resp. entire) function, $q \in \mathbb{C} \setminus \{0\}, \eta \in \mathbb{C}$. Let $P(z) = a_l z^l + a_{l-1} z^{l-1} + \cdots + a_1 z + a_0$ is a nonzero polynomial, where $a_0, a_1, \ldots, a_l (\neq 0)$ are complex constants, and *t* is the number of distinct zeros of *P(z)*. Then for $l > t + 4(resp. l > t)(l > t + 6(resp. l > t + 2))$, $P(f)f(qz+\eta) = a(z)(P(f)[f(qz+\eta) - f(z)] = a(z))$has infinitely many solutions, where $a(z) \in S(r, f)\backslash\{0\}$.

Motivated by Theorem A, Theorem B and q-difference Nevanlinna theory, we investigate the zero distribution of high order q-difference polynomials, and obtained the following theorem.

Theorem 1. Let *f(z)* be a transcendental meromorphic (resp.entire) function of zero order, $q \in \mathbb{C} \setminus \{0,1\}$ is a complex constant such that $\Delta_q^k f(z) \not\equiv 0$, *P(z)* and *t* are as in Theorem B, $a(z) \not\equiv (0, \infty)$ is a small function of *f(z)*. Then for $l \geq 3k + t + 4(l \geq t + 2)$, $P(f)\Delta_q^k f(z) - a(z)$ has infinitely many zeros.

Theorem 2. Let *f(z)* be a transcendental meromorphic (resp.entire) function of zero order, $q \in \mathbb{C} \setminus \{0,1\}$ is a complex constant such that $\Delta_q^k f(z) \not\equiv 0$, *P(z)* and *t* are as in Theorem B, $a(z) \not\equiv (0, \infty)$ is a small function of *f(z)*, if $n \geq k(l + t + 1) + t + l + 4(n \geq l + t + 2)$, then $f(z)^n P(\Delta_q^k f(z)) - a(z)$ has infinitely many zeros.

Remark 1. Theorem 1 is a generalization of Theorem B. It is easy to see that Theorem B is of case k=1.

The zero distribution of differential polynomials is a classical topic in the theory of meromorphic functions. Liu et al.[12] investigated the zero distribution of difference polynomials $[f(z)^n f(z+c)]^{(m)} - a(z)$ and $[f(z)^n \Delta_c f(z)]^{(m)} - a(z)$, where *a(z)* is a nonzero small function with respect to *f(z)*. Recently, Cao et al.[3] considered zeros of q-difference differential polynomials and obtain the following theorems.

Theorem C. Let *f(z)* be a transcendental meromorphic (resp.entire) function with zero order and *a(z)* is a nonzero small function with respect to *f(z)*. If $n \geq m + 6(n \geq m + 4)$, then $[f(z)^n f(qz+c)]^{(m)} - a(z)$ has infinitely many zeros.

Theorem D. Let *f(z)* be a transcendental meromorphic (resp.entire) function with zero order and *a(z)* is a nonzero small function with respect to *f(z)*, $f(qz+c) \neq f(z)$. If $n \geq m + 8(n \geq m + 4)$, then $[f(z)^n f(qz+c) - f(z)](m) - a(z)$ has infinitely many zeros.

In this paper, we will study the zero distribution of q-difference differential polynomials of the following form:

Theorem 3. Let *f(z)* be a transcendental meromorphic (resp.entire) function of zero order, $q \in \mathbb{C} \setminus \{0,1\}$ is a complex constant such that $\Delta_q^k f(z) \not\equiv 0$, *P(z)* and *t* are as in Theorem B, $a(z) \not\equiv (0, \infty)$ is a small function of f(z). Then for $l \geq 3k + m + t + 5(l \geq m + t + 3)$, $[P(f)\, \Delta_q^k f(z)]^{(m)} - a(z)$ has infinitely many zeros.

2. Some Lemmas

Lemma 1. [19] Let *f* be a meromorphic function in the complex plane, $a_i (i = 1,2,3)$ are distinct complex constants. Then

$$T(r,f) \leq \sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{f-a_i}\right) + S(r,f).$$

Lemma 2. [13] Let f be a non-constant zero order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$, then

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S_1(r,f) = o\big(T(r,f)\big)$$

on a set of lower logarithmic density 1.

Lemma 3. [21] Let f be a non-constant zero order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$, then

$$T(r, f(qz)) = (1+o(1))T(r,f)$$

on a set of lower logarithmic density 1.

Lemma 4. [21] Let f be a non-constant zero order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$, then

$$N(r, f(qz)) = (1+o(1))N(r,f)$$

on a set of lower logarithmic density 1.

Lemma 5. [18] Let f be a non-constant meromorphic function, and $P(z) = a_l z^l + a_{l-1} z^{l-1} + \cdots + a_1 z + a_0$, where $a_0, a_1, \ldots, a_l (\neq 0)$ are meromorphic functions and satisfies $T(r, a_i) = S(r,f) (i = 1,2,\ldots,l)$. Then

$$T(r, P(f)) = lT(r,f) + S(r,f).$$

Lemma 6. [10] Let f be a non-constant meromorphic function, s, m are positive integers, then

$$N_s\left(r, \frac{1}{f^{(m)}}\right) \leq T\big(r, f^{(m)}\big) - T(r,f) + N_{s+m}\left(r, \frac{1}{f}\right) + S(r,f),$$

$$N_s\left(r, \frac{1}{f^{(m)}}\right) \leq m\overline{N}(r,f) + N_{s+m}\left(r, \frac{1}{f}\right) + S(r,f),$$

where $N_s\left(r, \frac{1}{f}\right)$ denotes the counting function of the zeros of f, and the zeros of f with multiplicity k is counted k times if $k \leq s$ and s times if $k > s$. Obviously, $\overline{N}\left(r, \frac{1}{f}\right) = N_1\left(r, \frac{1}{f}\right)$.

Lemma 7. Let f be a transcendental meromorphic function of zero order, $q \in \mathbb{C}\backslash\{0,1\}$ is a complex constant such that $\Delta_q^k f(z) \not\equiv 0$, *P(z)* be stated as above. Then

$$(l-k)T(r,f) + S_1(r,f) \leq T(r, P(f)\Delta_q^k f) \leq (l+k+1)T(r,f) + S_1(r,f).$$

If f is a transcendental entire function of zero order, then we have

$$lT(r,f) + S_1(r,f) \leq T(r, P(f)\Delta_q^k f) \leq (l+1)T(r,f) + S_1(r,f).$$

Proof. Set $F(z) = P(f(z))\Delta_q^k f(z)$. If f is a transcendental meromorphic function of zero order, from the Valiron-Mohon'ko lemma and Lemma 2, we have

$$\begin{aligned} T(r,F) &\leq T(r,P(f)) + T\big(r, \Delta_q^k f\big) = lT(r,f) + m\big(r, \Delta_q^k f\big) + N\big(r, \Delta_q^k f\big) + S_1(r,f) \\ &\leq lT(r,f) + m\left(r, \frac{\Delta_q^k f}{f}\right) + m(r,f) + (k+1)N(r,f) + S_1(r,f) \\ &\leq lT(r,f) + T(r,f) + kN(r,f) + S_1(r,f) \leq (l+k+1)T(r,f) + S_1(r,f). \end{aligned}$$

On the other hand,

$$\begin{aligned} (l+1)T(r,f) = \; T(r, f^{l+1}) &\leq T\left(r, \frac{Ff}{\Delta_q^k f}\right) + S_1(r,f) \leq T(r,F) + T\left(r, \frac{f}{\Delta_q^k f}\right) + S_1(r,f) \\ &\leq T(r,F) + T\left(r, \frac{\Delta_q^k f}{f}\right) + O(1) + S_1(r,f) \leq T(r,F) + N\left(r, \frac{\Delta_q^k f}{f}\right) + S_1(r,f) \\ &\leq T(r,F) + N\left(r, \frac{1}{f}\right) + kN(r,f) + S_1(r,f) \leq T(r,F) + (k+1)T(r,f) + S_1(r,f). \end{aligned}$$

Then

$$(l-k)T(r,f)+S_1(r,f)\le T(r,P(f)\Delta_q^k f)\le (l+k+1)T(r,f)+S_1(r,f).$$

If f is a transcendental entire function of zero order, from Lemma 2, we have

$$\begin{aligned}T\big(r,P(f)\Delta_q^k f\big)=m\big(r,P(f)\Delta_q^k f\big)\le m(r,P(f)f)+m\left(r,\frac{\Delta_q^k f}{f}\right)\\ \le (l+1)m(r,f)+S_1(r,f)\\ =(l+1)T(r,f)+S_1(r,f).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(l+1)T(r,f)= T(r,f^{l+1})= m(r,f^{l+1})\le m\left(r,\frac{f^{l+1}}{F}\right)+ m(r,F)\le m\left(r,\frac{f^l}{P(f)}\right)+m\left(r,\frac{f}{\Delta_q^k f}\right)+T(r,F)\\ \le T(r,F)+T\left(r,\frac{\Delta_q^k f}{f}\right)+O(1)+S_1(r,f)\le T(r,F)+N\left(r,\frac{\Delta_q^k f}{f}\right)+S_1(r,f)\\ \le T(r,F)+N\left(r,\frac{1}{f}\right)+kN(r,f)+S_1(r,f)\le T(r,F)+T(r,f)+S_1(r,f).\end{aligned}$$

So

$$lT(r,f)+S_1(r,f)\le T(r,P(f)\Delta_q^k f)\le (l+1)T(r,f)+S_1(r,f).$$

Similar to Lemma 7, we have the following Lemma.

Lemma 8. Let f be a transcendental meromorphic function of zero order, $q\in\mathbb{C}\backslash\{0,1\}$ is a complex constant such that $\Delta_q^k f(z)\not\equiv 0$, *P(z)* be stated as above. Then

$$(n-kl-l)T(r,f)+S_1(r,f)\le T(r,f^nP(\Delta_q^k f))\le (n+kl+l)T(r,f)+S_1(r,f).$$

If f is a transcendental entire function of zero order, then we have

$$(n-l)T(r,f)+S_1(r,f)\le T(r,f^nP(\Delta_q^k f))\le (n+l)T(r,f)+S_1(r,f).$$

Lemma 9. Let f be a transcendental meromorphic function of zero order, $q\in\mathbb{C}\backslash\{0,1\}$ is a complex constant such that $\Delta_q^k f(z)\not\equiv 0$, *P(z)* be stated as above, $z_1,z_2,\dots,z_t$ are distinct zeros of *P(z)*. Then

$$\overline{N}\left(r,\frac{1}{P(f)\Delta_q^k f}\right)\le (t+1)T(r,f)+kN(r,f)+S_1(r,f).$$

Proof. By the First Fundamental Theorem and Lemma 3, we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{P(f)\Delta_q^k f}\right)\le \overline{N}\left(r,\frac{1}{P(f)}\right)+\overline{N}\left(r,\frac{1}{\Delta_q^k f}\right)\le\sum_{j=1}^{t}\overline{N}\left(r,\frac{1}{f-z_j}\right)+T\big(r,\Delta_q^k f\big)+O(1)\\ \le tT(r,f)+m\big(r,\Delta_q^k f\big)+N\big(r,\Delta_q^k f\big)+O(1)\\ \le tT(r,f)+m\left(r,\frac{\Delta_q^k f}{f}\right)+m(r,f)+(k+1)N(r,f)+O(1)\\ \le (t+1)T(r,f)+kN(r,f)+S_1(r,f).\end{aligned}$$

Lemma 10. Let f be a transcendental meromorphic function of zero order, $q\in\mathbb{C}\backslash\{0,1\}$ is a complex constant such that $\Delta_q^k f(z)\not\equiv 0$, *P(z)* be stated as above, $z_1,z_2,\dots,z_t$ are distinct zeros of *P(z)*. Then

$$\overline{N}\left(r,\frac{1}{f^nP(\Delta_q^k f)}\right)\le \overline{N}\left(r,\frac{1}{f}\right)+tkN(r,f)+tT(r,f)+S_1(r,f).$$

Proof. From the First Fundamental Theorem and Lemma 3, we have

$$\overline{N}\left(r,\frac{1}{f^nP(\Delta_q^kf)}\right)\leq\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{P(\Delta_q^kf)}\right)\leq\overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{t}\overline{N}\left(r,\frac{1}{\Delta_q^kf-z_j}\right)$$
$$\leq\overline{N}\left(r,\frac{1}{f}\right)+tT\left(r,\Delta_q^kf\right)+O(1)\leq\overline{N}\left(r,\frac{1}{f}\right)+tm\left(r,\Delta_q^kf\right)+tN\left(r,\Delta_q^kf\right)+O(1)$$
$$\leq\overline{N}\left(r,\frac{1}{f}\right)+tm\left(r,\frac{\Delta_q^kf}{f}\right)+tm(r,f)+t(k+1)N(r,f)+O(1)$$
$$\leq\overline{N}\left(r,\frac{1}{f}\right)+tkN(r,f)+tT(r,f)+S_1(r,f).$$

The proof is complete.

3. The Proof Of Theorem 1 and Theorem 2

Proof of Theorem 1. Set $F(z)=P(f)\Delta_q^kf$. From Lemma 7, we know that $T(r,F)=T(r,f)$ and $S_1(r,F)=$ $S1r,f$. Next we will consider the following two cases.

Case 1. If f is a transcendental meromorphic function of zero order, by Lemma 1 and Lemma 9, we have

$$T(r,F)\leq\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-a}\right)+S(r,F)$$
$$\leq\overline{N}(r,f)+\overline{N}\left(r,\Delta_q^kf(z)\right)+(t+1)T(r,f)+kN(r,f)+\overline{N}\left(r,\frac{1}{F-a}\right)+S_1(r,f)$$
$$\leq(2k+2)N(r,f)+(t+1)T(r,f)+\overline{N}\left(r,\frac{1}{F-a}\right)+S_1(r,f)$$
$$\leq(2k+t+3)T(r,f)+\overline{N}\left(r,\frac{1}{F-a}\right)+S_1(r,f).$$

From Lemma 7, we obtain

$$(l-k)T(r,f)+S_1(r,f)\leq(2k+t+3)T(r,f)+\overline{N}\left(r,\frac{1}{F-a}\right)+S_1(r,f).$$

Then $P(f)\Delta_q^kf-a(z)$ has infinitely many zeros as $l\geq 3k+t+4$.

Case 2. If f is a transcendental entire function. By using the same argument as in Case 1, we have

$$lT(r,f)+S_1(r,f)\leq(t+1)T(r,f)+\overline{N}\left(r,\frac{1}{F-a}\right)+S_1(r,f).$$

Then conclusion holds as $l\geq t+2$.

Proof of Theorem 2. Similar to the proof of Theorem 1, and using Lemma 7 and Lemma 9, we can prove Theorem 2 directly.

4. The Proof Of Theorem 3

Proof. Set $F(z)=P(f)\Delta_q^kf$. From Lemma 7, we know that $T(r,F^{(m)}=T(r,F)=T(r,f)$ and $S_1(r,F^{(m)}=$ $S_1(r,F)=S_1(r,f)$. Next, we will consider the following two cases.

Case 1. If f is a transcendental meromorphic function of zero order, we first suppose that $[P(f)\,\Delta_q^kf]^{(m)}-a(z)$ has finitely many zeros. By Lemma 1 and Lemma 6, we have

$$T\left(r,F^{(m)}\right)\leq\overline{N}\left(r,F^{(m)}\right)+\overline{N}\left(r,\frac{1}{F^{(m)}}\right)+\overline{N}\left(r,\frac{1}{F^{(m)}-a}\right)+S\left(r,F^{(m)}\right)$$
$$\leq\overline{N}(r,F)+T\left(r,F^{(m)}\right)-T(r,F)+N_{m+1}\left(r,\frac{1}{F}\right)+S\left(r,F^{(m)}\right).$$

Combining this inequality and Lemma 7, we obtain

$$
\begin{aligned}
(l-k)T(r,f)+S_1(r,f) &\le T(r,F) \le \overline{N}(r,F)+N_{m+1}\left(r,\frac{1}{F}\right)+S\left(r,F^{(m)}\right)\\
&\le \overline{N}(r,f)+\overline{N}\left(r,\Delta_q^k f\right)+N_{m+1}\left(r,\frac{1}{P(f)}\right)+N_{m+1}\left(r,\frac{1}{\Delta_q^k f}\right)+S_1(r,f)\\
&\le (k+2)\overline{N}(r,f)+\sum_{j=1}^{t}N_{m+1}\left(r,\frac{1}{f-z_j}\right)+N_{m+1}\left(r,\frac{f}{f\Delta_q^k f}\right)+S_1(r,f)\\
&\le (k+2)\overline{N}(r,f)+tT(r,f)+(m+1)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{f}{\Delta_q^k f}\right)+S_1(r,f)\\
&\le (k+2)\overline{N}(r,f)+tT(r,f)+(m+1)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{\Delta_q^k f}{f}\right)+m\left(r,\frac{\Delta_q^k f}{f}\right)+S_1(r,f)\\
&\le (k+2)\overline{N}(r,f)+tT(r,f)+(m+1)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+kN(r,f)+S_1(r,f)\\
&\le (m+2k+t+4)T(r,f)+S_1(r,f).
\end{aligned}
$$

By assumption, we get a contradiction for $l \ge m+3k+t+5$. Then $[P(f)\,\Delta_q^k f]^{(m)}-a(z)$ has infinitely many zeros.

Case 2. If f is a transcendental entire function, by the same method, we get

$$lT(r,f)+S_1(r,f) \le (m+t+2)T(r,f)+S_1(r,f),$$

which is a contradiction with $l \ge m+t+3$. Thus, we complete the proof of Theorem 3.

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