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Quasilinear fractional differential equation with resonance boundary condition*

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Abstract: In this paper, we consider the following quasilinear fractional differential equation with resonance boundary condition

$$\begin{cases} {}^C D_{0+}^{\beta}\phi_p({}^C D_{0+}^{\alpha}u(t)) = f\left(t,u(t),{}^C D_{0+}^{\alpha-1}u(t),{}^C D_{0+}^{\alpha}u(t)\right), t\in[0,1], \\ (\phi_p({}^C D_{0+}^{\alpha}u(0)))' = 0, \quad {}^C D_{0+}^{\alpha}u(\eta) = {}^C D_{0+}^{\alpha}u(1), \\ u'(0)=1, \quad u(\zeta)=u(1), \end{cases}$$

where ${}^C D_{0+}^{\alpha}$, ${}^C D_{0+}^{\beta}$ are Caputo fractional derivatives of order α, β, respectively, $1<\alpha\le 2$, $1<\beta\le 2$, $3<\alpha+\beta\le 4$, $\eta\in(0,1)$, $\zeta\in(0,1)$ and $p>1$, $\phi_p(s)=|s|^{p-2}s$ is a p-Laplacian operator, f is a continuous function. After translating the quasilinear equation into the linear fractional differential system, by using coincidence degree theory, the existence result is established.

Keywords: Fractional differential equation; p-Laplacian operator; Coincidence degree; Resonance.

1 Introduction

In this paper we will study the existence of solutions for the following quasilinear fractional differential equation with resonance boundary condition

$$\begin{cases} {}^C D_{0+}^{\beta}\phi_p({}^C D_{0+}^{\alpha}u(t)) = f(t,u(t),{}^C D_{0+}^{\alpha-1}u(t),{}^C D_{0+}^{\alpha}u(t)), t\in[0,1], \\ (\phi_p({}^C D_{0+}^{\alpha}u(0)))' = 0, \quad {}^C D_{0+}^{\alpha}u(\eta) = {}^C D_{0+}^{\alpha}u(1), \\ u'(0)=1, \quad u(\zeta)=u(1), \end{cases} \tag{1.1}$$

where ${}^C D_{0+}^{\alpha}$, ${}^C D_{0+}^{\beta}$ are Caputo fractional derivatives of order α, β, respectively, $1<\alpha\le 2$, $1<\beta\le 2$, $3<\alpha+\beta\le 4$, $\eta\in(0,1)$, $\zeta\in(0,1)$ and $p>1$, $\phi_p(s)=|s|^{p-2}s$ is a p-Laplacian operator, f is a continuous function.

In recent years, fractional differential equations have been of great of interest due to the intensive development of fractional calculus itself and its various applications. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, agriculture, etc. (see [4] [10] [11]).

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A broad range of scenarios of resonant problems were studied in the framework of ordinary differential and difference equations, (see [8] [16]). For fractional boundary value problems at resonance, we refer the reader to [1] [18] [19] and the references cited therein. Kosmatov (see [5]) studied the following boundary value problem of fractional order with non-local conditions

$$D^{\alpha}u(t) = f(t,u(t),u'(t)),\ a.e.\, t \in (0,1),$$
$$D_{0+}^{\alpha-2}u(0) = 0,\ \eta u(\xi) = u(1),$$

where $1<\alpha<2$, $0<\xi<1$ and $\eta\xi^{\alpha-1}=1$. This problem is resonance boundary value problem. The author obtained the existence result by the the coincidence degree theory of Mawhin.

p-Laplacian equations is very interesting because it has many applications. The turbulent flow in a porous problem medium is a fundamental mechanics problem. For studying this type of problems, Leibenson(see [6]) first introduced the p-Laplacian equation as follows

$$(\phi_p(x'(t)))' = f(t,x(t),x'(t)), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p>1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q>1$ is a constant such that $\frac{1}{p}+\frac{1}{q}=1$. From then on, many important results relative to (1.2) with certain boundary conditions had been obtained, (see [12] [13] [14] [15]). In [2], Chen studied the following boundary value problem for fractional p-Laplacian equation

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}x(t)) = f(t,x(t),D_{0+}^{\alpha}x(t)), t\in[0,1], \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) = 0, \end{cases}$$

where $0<\alpha,\beta\le 1$, $1<\alpha+\beta<2$, D_{0+}^{α}, D_{0+}^{β} is a Caputo fractional derivative, and $p>1$, $\phi_p(s)=|s|^{p-2}s$ is a p-Laplacian operator. A new result on the existence of the solutions for above fractional boundary value problem is obtained.

From the above references, we find that: for the resonance case, most of the BVPs considered are not more than second-order, and higher-order are restricted to the case $p=2$; most of the BVPs considered are related to Riemann-Liouville fractional derivative, the Caputo fractional derivative considered is less. Motivated by the works mentioned above, we study the existence of solutions for higher-order fractional boundary value problem with a p-Laplacian at resonance.

Because of the fact that the Mawhin's continuation theorem can't be used directly to discuss the BVP with a quasilinear differential operator, we translate the problem (1.1) into a system with linear differential operator. By the coincidence degree theorem of Mawhin, we obtain an existence result.

This paper is organized as follows: in section 2, we include some basic definitions and preliminary results that will be used to prove our main results; in section 3, using the coincidence degree theory of Mawhin(see [7]), we establish a theorem on existence of solutions for BVP (1.1); in section 4, an example is given to illustrate the main result.

2 Preliminaries and lemmas

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory, which can be found, for instance in [4] [11].

Definition 2.1 ([4] [11], section 2.1) The Riemann-Liouville fractional integral operator of order $\alpha>0$ of a function $u:(0,+\infty)\to R$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([4] [11], section 2.4) The Caputo fractional derivative of order $\alpha>0$ of a function $u:(0,+\infty)\to R$ is given by

$$ {}^{C}D_{0+}^{\alpha}u(t) = I_{0+}^{n-\alpha}\frac{d^n u(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds, $$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.1 ([4]) *Let* $\alpha > 0$. *Assume that* $u, {}^{C}D_{0+}^{\alpha}u \in L[0,1]$. *Then the following equality holds*

$$ I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, $$

where $c_i \in R, i = 0,1,\cdots,n-1$; here n is the smallest integer greater than or equal to α.

Proposition 2.2 ([3]) ϕ_p *satisfies the following properties*

(B1) ϕ_p *is continuous, monotonically increasing and invertible. Moreover,* $\phi_p^{-1} = \phi_q$ *with* $q > 1$ *satisfying* $\dfrac{1}{p} + \dfrac{1}{q} = 1$;

(B2) *for* $\forall\, u, v \geq 0$,

$$ \phi_p(u+v) \leq \phi_p(u) + \phi_p(v),\ \ if\ 1 < p < 2; $$

$$ \phi_p(u+v) \leq 2^{p-2}(\phi_p(u) + \phi_p(v)),\ \ if\ \ p \geq 2. $$

Definition 2.3 ([7]) Let X and Y be a real normed spaces. A linear mapping $L : \mathrm{dom}L \subset X \to Y$ is called a Fredholm mapping if the following two conditions holds:

(C1) $\ker L$ has a finite dimension, and

(C2) $\mathrm{Im}L$ is closed and has a finite codimension.

If L is a Fredholm mapping, its Fredholm index is the integer $\mathrm{Ind}L = \dim\ker L - \mathrm{codim}\,\mathrm{Im}L$.

Now, we briefly recall some notations, which can be found in [7]. Let X and Y be real Banach spaces, $L : \mathrm{dom}L \subset X \to Y$ be a Fredholm operator with index zero, and $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$ \mathrm{Im}P = \ker L,\quad \ker Q = \mathrm{Im}L, $$

$$ X = \ker L \oplus \ker P,\quad Y = \mathrm{Im}L \oplus \mathrm{Im}Q. $$

It follows that $L|_{\mathrm{dom}L \cap \ker P} : \mathrm{dom}L \cap \ker P \to \mathrm{Im}L$ is invertible. We denote the inverse by $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \ker P$.

If Ω is an open bounded subset of X such that $\mathrm{dom}L \cap \overline{\Omega} \neq \varnothing$, then the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.3 ([7], Theorem IV.13) *Let* X *and* Y *be two Banach spaces with norms* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$, *respectively, and* $\Omega \subset X$ *an open and bounded set. Suppose* $L : X \cap \mathrm{dom}L \to Y$ *is a Fredholm operator of index zero and* $N_\lambda : \overline{\Omega} \to Y, \lambda \in [0,1]$ *is L-compact. In addition, if*

(D1) $Lx \neq \lambda Nx$ for $\lambda \in (0,1)$, $x \in (\mathrm{dom}L \setminus \ker L) \cap \partial\Omega$;

(D2) $Nx \notin \mathrm{Im}L$ for $x \in \ker L \cap \partial\Omega$;

(D3) $\deg\{JQN|_{\overline{\Omega} \cap \ker L}, \Omega \cap \ker L, 0\} \neq 0$, where $Q : Y \to Y$ is a projection such that $\mathrm{Im}L = \ker Q$

and $J:\mathrm{Im}Q \to \ker L$ is a homeomorphism.

Then the abstract equation $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

Let $x_1(t) = u(t)$, $x_2(t) = \phi_p({}^C D_{0+}^{\alpha} u(t))$. Rewrite the differential equation in BVP (1.1) into

$$\begin{cases} {}^C D_{0+}^{\alpha} x_1(t) = \phi_q(x_2(t)), & (2.1) \\ {}^C D_{0+}^{\beta} x_2(t) = f(t, x_1(t), {}^C D_{0+}^{\alpha-1} x_1(t), \phi_q(x_2(t))), & (2.2) \\ x_1'(0) = 0, \quad x_1(\zeta) = x_1(1), & (2.3) \\ x_2'(0) = 0, \quad x_2(\eta) = x_2(1). & (2.4) \end{cases}$$

In this paper, we take $Z_1 = \{z_1 \mid z_1, {}^C D_{0+}^{\alpha-1} z_1, {}^C D_{0+}^{\alpha} z_1 \in C[0,1]\}$ with norm $\|z_1\|_{Z_1} = \max\{\|z_1\|_\infty, \|{}^C D_{0+}^{\alpha-1} z_1\|_\infty, \|{}^C D_{0+}^{\alpha} z_1\|_\infty\}$, $Z_2 = \{z_2 \mid z_2, {}^C D_{0+}^{\beta-1} z_2, {}^C D_{0+}^{\beta} z_2 \in C[0,1]\}$ with norm $\|z_2\|_{Z_2} = \max\{\|z_2\|_\infty, \|{}^C D_{0+}^{\beta-1} z_2\|_\infty, \|{}^C D_{0+}^{\beta} z_2\|_\infty\}$.

Now we set $X = \{x = (x_1, x_2)^{\mathrm{T}} \in Z_1 \times Z_2\}$ with the norm $\|x\|_X = \max\{\|x_1\|_{Z_1}, \|x_2\|_{Z_2}\}$, let $Y = \{y = (y_1, y_2)^{\mathrm{T}} \in C[0,1] \times C[0,1]\}$ with norm $\|y\|_Y = \max\{\|y_1\|_\infty, \|y_2\|_\infty\}$. Clearly, X and Y are Banach spaces.

Define $L:\mathrm{dom}L \to Y$ by

$$Lx = L(x_1, x_2)^{\mathrm{T}} = ({}^C D_{0+}^{\alpha} x_1, {}^C D_{0+}^{\beta} x_2)^{\mathrm{T}}, \tag{2.5}$$

where

$$\mathrm{dom}L = \{x = (x_1, x_2)^{\mathrm{T}} \in X : x_1'(0) = 0, x_1(\zeta) = x_1(1), x_2'(0) = 0, x_2(\eta) = x_2(1)\}. \tag{2.6}$$

Obviously, if $x = (x_1, x_2)^{\mathrm{T}} \in \mathrm{dom}L$ is a solution of (2)-(2), then x_1 is a solution of BVP (1.1).

3 Main results

In this section, a theorem on existence of solution for BVP (1.1) will be given.

Theorem 3.1 *Suppose*

(H1) there exists a constant $A > 0$ such that

$$\frac{\Gamma(\alpha+1)}{1-\zeta^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q(x_2(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{\zeta} (\zeta-s)^{\alpha-1}\phi_q(x_2(s))ds\right) \neq 0, \tag{3.1}$$

for $x \in \mathrm{dom}L \setminus \ker L$ with $|x_2(t)| > A$ on $t \in [0,1]$;

(H2) there exists a constant $B > 0$ such that

$$\frac{\Gamma(\beta+1)}{1-\eta^{\beta}}\left(\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f\left(s, x_1(s), {}^C D_{0+}^{\alpha-1} x_1(s), \frac{1}{\lambda}\left({}^C D_{0+}^{\alpha} x_1(s)\right)\right)ds - \right.$$
$$\left. \frac{1}{\Gamma(\beta)}\int_0^{\eta} (\eta-s)^{\beta-1} f\left(s, x_1(s), {}^C D_{0+}^{\alpha-1} x_1(s), \frac{1}{\lambda}\left({}^C D_{0+}^{\alpha} x_1(s)\right)\right)ds\right) \neq 0. \tag{3.2}$$

for $x \in \mathrm{dom}L \setminus \ker L$ with $|x_1(t)| > B$ on $t \in [0,1]$, where $\lambda \in (0,1)$;

(H3) there exists function $\mu, \nu, \gamma, \rho, \in C([0,1], R^{+})$ such that for $\forall(x,y,z)\in R^3$ and $t\in[0,1]$,

$$|f(t,x,y,z)| \le \mu(t)|x|^{p-1} + \nu(t)|y|^{p-1} + \gamma(t)|z|^{p-1} + \rho(t), \tag{3.3}$$

we denote that $\mu_0 = \|\mu\|_\infty$, $\nu_0 = \|\nu\|_\infty$, $\gamma_0 = \|\gamma\|_\infty$, $\rho_0 = \|\rho\|_\infty$.

Then BVP (1.1) has at least one solution provided

$$\frac{1}{\Gamma(\beta)}\left(\frac{1}{\Gamma(\alpha)^{p-1}}\mu_0 + \nu_0 + \gamma_0\right) < 1, \ if \ p < 2; \tag{3.4}$$

$$\frac{1}{\Gamma(\beta)}\left(\frac{2^{p-2}}{\Gamma(\alpha)^{p-1}}\mu_0 + \nu_0 + \gamma_0\right) < 1, \ if \ p \ge 2. \tag{3.5}$$

Now, we begin with some lemmas below.

Lemma 3.2 *Let L be defined by (2.1); then*

$$\ker L = \{x = (c_1, c_2)^{\mathrm{T}} \in \mathrm{dom}L : c_1, c_2 \in R\}, \tag{3.6}$$

$$\mathrm{Im}L = \{y = (y_1, y_2)^{\mathrm{T}} \in Y : I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_1(\zeta) = 0, I_{0+}^{\beta} y_2(1) - I_{0+}^{\beta} y_2(\eta) = 0\}. \tag{3.7}$$

Proof. First we show (3.6). By Lemma 2.1, ${}^{C}D_{0+}^{\alpha} x_1(t) = 0$ has a solution

$$x_1(t) = c_1 + c_0 t, \, c_0, c_1 \in R,$$

Combining with boundary value condition $x_1'(0) = 0$, one has $x_1(t) = c_1 \in R$. Similarly from ${}^{C}D_{0+}^{\beta} x_2(t) = 0$, we have $x_2(t) = c_2 \in R$. One has that (3.6) holds.

For $x = (x_1, x_2)^{\mathrm{T}} \in \mathrm{dom}L$, consider the system

$$\begin{cases} {}^{C}D_{0+}^{\alpha} x_1(t) = y_1(t), & (3.8) \\ {}^{C}D_{0+}^{\beta} x_2(t) = y_2(t). & (3.9) \end{cases}$$

It holds that $y = (y_1, y_2)^{\mathrm{T}} \in Y$. From (3) and (2.2), using Lemma 2.1, we can get

$$I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_1(\zeta) = 0, \tag{3.10}$$

Also, in view of (3) and (2.2), we have

$$I_{0+}^{\beta} y_2(1) - I_{0+}^{\beta} y_2(\eta) = 0. \tag{3.11}$$

Thus

$$\mathrm{Im}L \subset \{y = (y_1, y_2)^{\mathrm{T}} \in Y : I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_1(\zeta) = 0, I_{0+}^{\beta} y_2(1) - I_{0+}^{\beta} y_2(\eta) = 0\}. \tag{3.12}$$

Conversely, we can show that $\{y = (y_1, y_2)^{\mathrm{T}} \in Y : I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_1(\zeta) = 0, I_{0+}^{\beta} y_2(1) - I_{0+}^{\beta} y_2(\eta) = 0\} \subset \mathrm{Im}L$. Hence

$$\mathrm{Im}L = \{y = (y_1, y_2)^{\mathrm{T}} \in Y : I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_1(\zeta) = 0, I_{0+}^{\beta} y_2(1) - I_{0+}^{\beta} y_2(\eta) = 0\}. \tag{3.13}$$

Lemma 3.3 *Let L be defined by (2.1); then L is a Fredholm operator of index zero, and the linear continuous projector operators $P: X \to \ker L$ and $Q: Y \to \mathrm{Im}Q$ can be defined as*

$$Px = (x_1(0), x_2(0))^{\mathrm{T}}, \tag{3.14}$$

$$Qy = \left(\frac{\Gamma(\alpha+1)}{1-\zeta^{\alpha}} (I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_1(\zeta)), \frac{\Gamma(\eta+1)}{1-\eta^{\beta}} (I_{0+}^{\beta} y_2(1) - I_{0+}^{\beta} y_2(\eta)) \right)^{\mathrm{T}}. \tag{3.15}$$

Let $L_P = L|_{\mathrm{dom}L \cap \ker P}$ and $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \ker P$ denote the inverse of L_P. Set

$$K_P y(t) = K_P (y_1(t), y_2(t))^{\mathrm{T}} = \left(I_{0+}^{\alpha} y_1(t), I_{0+}^{\beta} y_2(t) \right)^{\mathrm{T}}. \tag{3.16}$$

Proof. For any $y \in Y$, we have

$$Q^2 y = Q(Qy) = Q\left(\frac{\Gamma(\alpha+1)}{1-\zeta^{\alpha}} (I_{0+}^{\alpha} y_1(1) - I_{0+}^{\alpha} y_1(\zeta)), \frac{\Gamma(\eta+1)}{1-\eta^{\beta}} (I_{0+}^{\beta} y_2(1) - I_{0+}^{\beta} y_2(\eta)) \right)^{\mathrm{T}}$$

$$= Qy(\frac{\Gamma(\alpha+1)}{1-\zeta^{\alpha}} \left(\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds - \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} (\zeta - s)^{\alpha-1} ds \right),$$

$$\frac{\Gamma(\beta+1)}{1-\eta^{\beta}} \left(\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} ds - \frac{1}{\Gamma(\beta)} \int_0^{\eta} (\eta - s)^{\beta-1} ds \right))^{\mathrm{T}}$$

$$= Qy(1,1)^{\mathrm{T}} = Qy.$$

Moreover, (3.7) and (3.13) imply that $\mathrm{Im}L = \ker Q$, then $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$, $\mathrm{codim}\,\mathrm{Im}L = \dim \mathrm{Im}Q = 2 = \dim \ker L$. Hence, L is a Fredholm operator of index zero.

From the definitions of P, K_P, it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in \mathrm{Im}L$, we have

$$LK_P y(t) = L(I_{0+}^{\alpha} y_1(t), I_{0+}^{\beta} y_2(t))^{\mathrm{T}} = ({}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} y_1(t), {}^{C}D_{0+}^{\beta} I_{0+}^{\beta} y_2(t))^{\mathrm{T}}$$

$$= (y_1(t), y_2(t))^{\mathrm{T}} = y(t), \tag{3.17}$$

Moreover, for $x \in \mathrm{dom}L \cap \ker P,$ we get $x = (x_1(t), x_2(t))^{\mathrm{T}} = (0,0)^{\mathrm{T}}.$ By Lemma 2.1, we obtain that

$$K_P L x(t) = K_P ({}^{C}D_{0+}^{\alpha} x_1(t), {}^{C}D_{0+}^{\beta} x_2(t))^{\mathrm{T}} = (I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} x_1(t), I_{0+}^{\beta}\, {}^{C}D_{0+}^{\beta} x_2(t))^{\mathrm{T}}$$

$$= (x_1(t), x_2(t))^{\mathrm{T}} = x(t). \tag{3.18}$$

Combining (3.17) with (3.18), we know that $K_P = (L|_{\mathrm{dom}L \cap \ker L})^{-1}.$ The proof is complete.

Define $N : X \to Y$ by

$$Nx(t) = N(x_1(t), x_2(t))^{\mathrm{T}} = \left(\phi_q(x_2(t)), f\left(t, x_1(t), {}^{C}D_{0+}^{\alpha-1} x_1(t), \phi_q(x_2(t))\right) \right)^{\mathrm{T}}, \tag{3.19}$$

then (2)-(2) can be written as $Lx = Nx$.

Since f is a continuous function and $\phi_p(s)$ is a uniformly continuity function, we can prove by standard arguments that N is L-compact, i.e., QN and $K_p(I-Q)N$ are completely continuous.

Lemma 3.4 *Suppose* (H1)-(H3) *hold; then the set* $\Omega_1 = \{x \in \mathrm{dom}L \setminus \ker L : Lx = \lambda Nx, \lambda \in (0,1)\}$ *is bounded.*

Proof. Take $x \in \Omega_1$, then $\lambda Nx = Lx \in \mathrm{Im}L = \ker Q$. So $QNx = 0$, then

$$\frac{\Gamma(\alpha+1)}{1-\zeta^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\phi_q(x_2(s))ds-\frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-s)^{\alpha-1}\phi_q(x_2(s))ds\right)=0, \tag{3.20}$$

$$\frac{\Gamma(\beta+1)}{1-\eta^{\beta}}\left(\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f\left(s,x_1(s),{}^CD_{0+}^{\alpha-1}x_1(s),\phi_q(x_2(s))\right)ds-\right.$$
$$\left.\frac{1}{\Gamma(\beta)}\int_0^{\eta}(\eta-s)^{\beta-1}f\left(s,x_1(s),{}^CD_{0+}^{\alpha-1}x_1(s),\phi_q(x_2(s))\right)ds\right)=0. \tag{3.21}$$

It follows from (H1) and (3.18) that there exists $t_2\in[0,1]$ such that $|x_2(t_2)|\le A$. Now

$$|x_2(t)|=\left|x_2(t_2)+\int_{t_2}^t x_2'(s)ds\right|$$
$$\le|x_2(t_2)|+\left|\int_{t_2}^t x_2'(s)ds\right|$$
$$\le A+\|x_2'\|_\infty,\ \ \forall\, t\in[0,1],$$

thus, we get

$$\|x_2\|_\infty\le A+\|x_2'\|_\infty. \tag{3.22}$$

Using Lemma 2.1 and $x_2'(0)=0$, we have

$$|x_2'(t)|=|x_2'(0)+I_{0+}^{\beta-1}\,{}^CD_{0+}^{\beta}x_2(t)|$$
$$=\left|\frac{1}{\Gamma(\beta-1)}\int_0^t(t-s)^{\beta-2}\,{}^CD_{0+}^{\beta}x_2(s)ds\right|$$
$$\le\|{}^CD_{0+}^{\beta}x_2\|_\infty\left|\frac{1}{\Gamma(\beta-1)}\int_0^t(t-s)^{\beta-2}ds\right|$$
$$\le\frac{1}{\Gamma(\beta)}\|{}^CD_{0+}^{\beta}x_2\|_\infty,\ \ \forall\, t\in[0,1],$$

thus, we get

$$\|x_2'\|_\infty\le\frac{1}{\Gamma(\beta)}\|{}^CD_{0+}^{\beta}x_2\|_\infty. \tag{3.23}$$

Combining (3.20) with (3.21), we have

$$\|x_2\|_\infty\le A+\|x_2'\|_\infty\le A+\frac{1}{\Gamma(\beta)}\|{}^CD_{0+}^{\beta}x_2\|_\infty. \tag{3.24}$$

Using Lemma 2.1, we have

$$|{}^CD_{0+}^{\beta-1}x_2(t)|=|{}^CD_{0+}^{\beta-1}x_2(0)+I_{0+}^{1}\,{}^CD_{0+}^{\beta}x_2(t)|$$
$$=\left|\int_0^t{}^CD_{0+}^{\beta}x_2(s)ds\right|\le\|{}^CD_{0+}^{\beta}x_2\|_\infty,\ \ \forall\, t\in[0,1].$$

thus, we get

$$\|{}^{C}D_{0+}^{\beta-1}x_2\|_\infty \le \|{}^{C}D_{0+}^{\beta}x_2\|_\infty. \tag{3.25}$$

If $x \in \Omega_1$, then

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x_1(t) = \lambda\phi_q(x_2(t)), & (3.26)\\ {}^{C}D_{0+}^{\beta}x_2(t) = \lambda f(t, x_1(t), {}^{C}D_{0+}^{\alpha-1}x_1(t), \phi_q(x_2(t))). & (3.27)\end{cases}$$

Substituting $x_2(t) = \phi_p\left(\frac{1}{\lambda}\left({}^{C}D_{0+}^{\alpha}x_1(t)\right)\right)$ into (3.21), we have

$$\frac{\Gamma(\beta+1)}{1-\eta^\beta}\left(\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1} f\left(s, x_1(s), {}^{C}D_{0+}^{\alpha-1}x_1(s), \frac{1}{\lambda}\left({}^{C}D_{0+}^{\alpha}x_1(t)\right)\right)ds - \right.$$

$$\left.\frac{1}{\Gamma(\beta)}\int_0^\eta(\eta-s)^{\beta-1} f\left(s, x_1(s), {}^{C}D_{0+}^{\alpha-1}x_1(s), \frac{1}{\lambda}\left({}^{C}D_{0+}^{\alpha}x_1(t)\right)\right)ds\right) = 0. \tag{3.28}$$

In view of (H2) and (3.21), we have there exists $t_1 \in [0,1]$. So $|x_1(t_1)| \le B$.

Similarly, we have

$$\|{}^{C}D_{0+}^{\alpha}x_1\|_\infty \le \|x_2\|_\infty^{q-1}, \tag{3.29}$$

$$\|{}^{C}D_{0+}^{\alpha-1}x_1\|_\infty \le \|{}^{C}D_{0+}^{\alpha}x_1\|_\infty \le \|x_2\|_\infty^{q-1}, \tag{3.30}$$

$$\|x_1\|_\infty \le B + \frac{1}{\Gamma(\alpha)}\|{}^{C}D_{0+}^{\alpha}x_1\|_\infty \le B + \frac{1}{\Gamma(\alpha)}\|x_2\|_\infty^{q-1} \tag{3.31}$$

(I) For $1 < p < 2$, from (H3) and Proposition 2.2 one gets

$$\|{}^{C}D_{0+}^{\beta}x_2\|_\infty = \max_{t\in[0,1]} |\lambda f(t, x_1(t), {}^{C}D_{0+}^{\alpha-1}x_1(t), \phi_q(x_2(t)))|$$

$$\le \max_{t\in[0,1]}\left(\mu(t)|x_1(t)|^{p-1} + \nu(t)|{}^{C}D_{0+}^{\alpha-1}x_1(t)|^{p-1} + \gamma(t)|\phi_q(x_2(t))|^{p-1} + \rho(t)\right)$$

$$\le \mu_0\|x_1\|_\infty^{p-1} + \nu_0\|{}^{C}D_{0+}^{\alpha-1}x_1\|_\infty^{p-1} + \gamma_0\|x_2\|_\infty + \rho_0$$

$$\le \mu_0\left(B + \frac{1}{\Gamma(\alpha)}\|x_2\|_\infty^{q-1}\right)^{p-1} + \nu_0(\|x_2\|_\infty^{q-1})^{p-1} + \gamma_0\|x_2\|_\infty + \rho_0$$

$$\le \left(\frac{1}{\Gamma(\alpha)^{p-1}}\mu_0 + \nu_0 + \gamma_0\right)\|x_2\|_\infty + \rho_0$$

$$\le \left(\frac{1}{\Gamma(\alpha)^{p-1}}\mu_0 + \nu_0 + \gamma_0\right)\left(A + \frac{1}{\Gamma(\beta)}\|{}^{C}D_{0+}^{\beta}x_2\|_\infty\right) + (\rho_0 + \mu_0 B^{p-1}).$$

Notice (3.4), one arrives at

$$\|{}^{C}D_{0+}^{\beta}x_2\|_{\infty} \le \frac{\left(\frac{1}{\Gamma(\alpha)^{p-1}}\mu_0+\nu_0+\gamma_0\right)A+\mu_0 B^{p-1}+\rho_0}{1-\frac{1}{\Gamma(\beta)}\left(\frac{1}{\Gamma(\alpha)^{p-1}}\mu_0+\nu_0+\gamma_0\right)} := M_1, \tag{3.31}$$

which yields $\|x_2\|_{\infty} \le A+\frac{1}{\Gamma(\beta)}M_1$, and then $\|x_2\|_{Z_2} = \max\{M_1, A+\frac{1}{\Gamma(\beta)}M_1\} := M_2$,

$$\|x_1\|_{Z_1} \le \max\{(A+\frac{1}{\Gamma(\beta)}M_1)^{q-1}, B+\frac{1}{\Gamma(\alpha)}(A+\frac{1}{\Gamma(\beta)}M_1)^{q-1}\} := M_3.$$

(II) For $p \ge 2$, similarly,

$$\|{}^{C}D_{0+}^{\beta}x_2\|_{\infty} \le \mu_0\|x_1\|_{\infty}^{p-1}+\nu_0\|{}^{C}D_{0+}^{\alpha}x_1\|_{\infty}^{p-1}+\gamma_0\|x_2\|_{\infty}+\rho_0$$

$$\le \mu_0 2^{p-2}\left(B^{p-1}+\frac{1}{\Gamma(\alpha)^{p-1}}\|x_2\|_{\infty}\right)+\nu_0\|x_2\|_{\infty}+\gamma_0\|x_2\|_{\infty}+\rho_0$$

$$\le \left(\frac{2^{p-2}}{\Gamma(\alpha)^{p-1}}\mu_0+\nu_0+\gamma_0\right)\|x_2\|_{\infty}+\mu_0 2^{p-2}B^{p-1}+\rho_0$$

$$\le \left(\frac{2^{p-2}}{\Gamma(\alpha)^{p-1}}\mu_0+\nu_0+\gamma_0\right)\left(A+\frac{1}{\Gamma(\beta)}\|{}^{C}D_{0+}^{\beta}x_2\|_{\infty}\right)+\mu_0 2^{p-2}B^{p-1}+\rho_0.$$

From (3.5), we have

$$\|{}^{C}D_{0+}^{\beta}x_2\|_{\infty} \le \frac{\left(\frac{2^{p-2}}{\Gamma(\alpha)^{p-1}}\mu_0+\nu_0+\gamma_0\right)A+\mu_0 2^{p-2}B^{p-1}+\rho_0}{1-\frac{1}{\Gamma(\beta)}\left(\frac{2^{p-2}}{\Gamma(\alpha)^{p-1}}\mu_0+\nu_0+\gamma_0\right)} := N_1 \tag{3.32}$$

which leads to $\|x_2\|_{Z_2} \le \max\{N_1, A+\frac{1}{\Gamma(\beta)}N_1\} := N_2$,

$\|x_1\|_{Z_1} \le \max\{(A+\frac{1}{\beta}N_1)^{q-1}, B+\frac{1}{\Gamma(\alpha)}(A+\frac{1}{\Gamma(\beta)}N_1)^{q-1}\} := N_3$. Thus,

$$\|x_1\|_{Z_1} \le \max\{M_3, N_3\} := M, \quad \|x_2\|_{Z_2} \le \max\{M_2, N_2\} := N.$$

$$\|x\|_X = \max\{\|x_1\|_{Z_1}, \|x_2\|_{Z_2}\} = \max\{M, N\}.$$

Therefore, Ω_1 is bounded. The proof is complete.

Lemma 3.5 *Suppose that* (H2) *holds, then the set* $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$ *is bounded.*

Proof. For $\forall x \in \Omega_2$, then $x = (c_1, c_2)^{\mathrm{T}}$ and

$$QNx=\left(\frac{\alpha}{1-\zeta^{\alpha}}\left(\int_0^1(1-s)^{\alpha-1}\phi_q(c_2)ds-\int_0^{\zeta}(\zeta-s)^{\alpha-1}\phi_q(c_2)ds\right),\right.$$

$$\left.\frac{\beta}{1-\eta^{\beta}}\left(\int_0^1(1-s)^{\beta-1}f\left(s,c_1,0,\phi_q(c_2)\right)ds-\int_0^{\eta}(\eta-s)^{\beta-1}f\left(s,c_1,0,\phi_q(c_2)\right)ds\right)\right)^{\mathrm{T}}$$

$$=\left(\phi_q(c_2),\frac{\beta}{1-\eta^{\beta}}\left(\int_0^1(1-s)^{\beta-1}f\left(s,c_1,0,\phi_q(c_2)\right)ds-\right.\right.$$

$$\left.\left.\int_0^{\eta}(\eta-s)^{\beta-1}f\left(s,c_1,0,\phi_q(c_2)\right)ds\right)\right)^{\mathrm{T}}$$

$$=(0,0)^{\mathrm{T}}.$$

So $c_2=0$. From (H2), we have $|c_1|\leq B$. Thus $||x||_X=|c_1|\leq B\leq M$, which implies $\Omega_2\subset\Omega_1$ is bounded.

Let $\Omega=\{x=(x_1,x_2)^{\mathrm{T}}\in \mathrm{dom}L:||x_1||_{Z_1}\leq M+1,||x_2||_{Z_2}\leq N+1\}$, then $\Omega\supset\overline{\Omega}_1\supset\overline{\Omega}_2$ is bounded and open set. Clearly, conditions (D1) and (D2) in Theorem 2.3 are satisfied. The remainder is to verify condition (D3). To this end, we define isomorphism $J:\mathrm{Im}Q\to\ker L$ by $J(x_1,x_2)^{\mathrm{T}}=(x_2,x_1)^{\mathrm{T}}$. Let $H(x,\lambda)=\lambda x+(1-\lambda)JQNx,\ \forall(x,\lambda)\in\overline{\Omega}\times[0,1]$. Then

$$H(x,\lambda)=\left(\lambda x_1+(1-\lambda)\left(\frac{\beta}{1-\eta^{\beta}}\left(\int_0^1(1-s)^{\beta-1}f\left(s,x_1(s),{}^CD_{0+}^{\alpha-1}x_1(s),\phi_q(x_2(s))\right)ds-\right.\right.\right.$$

$$\left.\left.\int_0^{\eta}(\eta-s)^{\beta-1}f\left(s,x_1(s),{}^CD_{0+}^{\alpha-1}x_1(s),\phi_q(x_2(s))\right)ds\right)\right),$$

$$\left.\lambda x_2+(1-\lambda)\left(\frac{\alpha}{1-\zeta^{\alpha}}\left(\int_0^1(1-s)^{\alpha-1}\phi_q(x_2(s))ds-\int_0^{\zeta}(\zeta-s)^{\alpha-1}\phi_q(x_2(s))ds\right)\right)\right)^{\mathrm{T}},$$

It is easy to see that $H(x,\lambda)\neq 0$ for $\forall(x,\lambda)\in(\partial\Omega\cap\ker L)\times[0,1]$. Hence,

$$\deg\{JQN|_{\overline{\Omega}\cap\ker L},\Omega\cap\ker L,0\}=\deg\{H(\cdot,0),\Omega\cap\ker L,0\}$$

$$=\deg\{H(\cdot,1),\Omega\cap\ker L,0\}$$

$$=\deg\{I,\Omega\cap\ker L,0\}\neq 0.$$

Theorem 2.3 yields that $Lx=Nx$ has at least one solution $x\in\mathrm{dom}L\cap\overline{\Omega}$. Namely, BVP (1.1) has at least one solution in X. The proof is complete.

4 Example

In this section, we give some examples to illustrate the usefulness of our main result.

Example 4.1

$$\begin{cases} {}^CD_{0+}^{1.5}\phi_3({}^CD_{0+}^{1.7}u(t))=f(t,u(t),{}^CD_{0+}^{0.7}u(t),{}^CD_{0+}^{1.7}u(t)),t\in[0,1],\\ (\phi_p({}^CD_{0+}^{1.7}u(0)))'=0,\quad {}^CD_{0+}^{1.7}u(0.25)={}^CD_{0+}^{1.7}u(1),\\ u'(0)=1,\quad u(0.5)=u(1), \end{cases}\tag{4.1}$$

Corresponding to BVP (1.1), we have $p=3$, $\alpha=1.7$, $\beta=1.5$, $\eta=0.25$, $\zeta=0.5$,

$$f(t,x,y,z)=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{8}+t-t^2\right)x^2+\frac{1}{8}t^2y^2+\frac{1}{16}tz^2,$$

Clearly, assumtions (H1)-(H2) are all satisfied. Let $\mu(t)=\frac{1}{2}\left(\frac{1}{4}+t-t^2\right)$, $\nu(t)=\frac{1}{8}t^2$, $\gamma(t)=\frac{1}{16}t$, $\rho(t)=\frac{1}{2}$, then $\mu_0=\frac{3}{16}$, $\nu_0=\frac{1}{8}$, $\gamma_0=\frac{1}{16}$, $\rho_0=\frac{1}{2}$,

$$\frac{1}{\Gamma(1.5)}\left(\frac{2}{\Gamma(1.7)^2}\frac{3}{16}+\frac{1}{8}+\frac{1}{16}\right)\approx 0.7241<1.$$

Then (H3) and (3.5) hold.

Therefore, BVP (1.1) has a solution by Theorem 3.1.

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