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Some q-Hypergeometric representations of the multiple Hurwitz zeta function

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Abstract

In this paper, The main object to give some new representations of the q-analogue of the multiple Hurwitz zeta function are derived.

Keywords: Multiple Hurwitz zeta function; q-Hypergeometric series; q-shifted factorial and special function.

1. Introduction, definitions and notations

The Hurwitz or generalized zeta function at integer points[13]

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \quad , \ 0<a\le 1, \tag{1.1}$$

has a q-analogue defined by

$$\zeta_q(s,a)=\sum_{n=0}^{\infty}\frac{q^{(n+a)(s-1)}}{[n+a]^s} \quad , \ 0<q<1, \ 0<a\le 1. \tag{1.2}$$

The series (1.2) is convergent for $\operatorname{Re} s>1$.

In [14] the q-analogue of Hurwitz zeta function is defined as

$$\zeta_q(s,a)=q^{a(s-1)}\{a\}_q^{-s}\sum_{n=0}^{\infty}q^{-n}\left[\frac{\langle a;q\rangle_n}{\langle a+1;q\rangle_n}q^n\right]^s \quad . \tag{1.3}$$

Barnes [4] (see also[1,2,3]) introduced and studied the generalized multiple Hurwitz zeta function $\zeta_n(s,a/w_1,\ldots,w_n)$ defined, for $R(s)>n$, by the following series:

$$\zeta_n(s,a/w_1,\ldots,w_n)=\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{1}{(a+\Omega)^s} \qquad (R(s))>n\ ;n\in N\ , \tag{1.4}$$

where N denotes the set of positive integers $\Omega=k_1w_1+\ldots+k_nw_n$.

Barnes-Changhee multiple q- zeta functions are defined by (see[9],[10]).

$$\zeta_{q,n}(s,w/a_1,a_2,\ldots,a_r)=\sum_{n_1,\ldots,n_n=0}^{\infty}\frac{q^{w+n_1+n_2+\ldots+n_r}}{(w+n_1a_1+n_2a_2+\ldots+n_ra_r)^s} \tag{1.5}$$

$R(w)>0,\ q\in C$ with $|q|<1$, which , for $a_1=a_2=\ldots=a_r=1$, yields

$$\zeta_{q,n}(s,w/1,1,\ldots,1)=\sum_{n_1,\ldots,n_n=0}^{\infty}\frac{q^{w+n_1+n_2+\ldots+n_r}}{(w+n_1+n_2+\ldots+n_r)^s} .$$

Moreover, if $w=r$ and $s=1-n\quad(n\in Z^+)$, we have

$$\zeta_{q,n}(n-1,r/1,1,\ldots,1)=(-1)^r\frac{(n-1)!}{(n+r-1)!}B_{n+r-1}^{(r)}(r;q),$$

where $B_{n+r-1}^{(r)}(r;q)$ is called q-Bernoulli numbers.

We also note that

$$\lim_{q\to1}\zeta_{q,n}(n-1,r/1,1,\ldots,1)=\zeta_n(n-1,r/1,1,\ldots,1)=(-1)^r\frac{(n-1)!}{(n+r-1)!}B_{n+r-1}^{(r)}(r)$$

(see[9],[10]).

Where the q-number $[z]_q$ is defined through

$$[z]_q=\frac{1-q^z}{1-q}\quad ,z\in C\ ,q\neq1. \tag{1.6}$$

A special case of (1.6) when $z\in N$ is $[n]_q=\dfrac{1-q^n}{1-q}=\sum_{0\le i\le n-1}q^i\quad ,n\in N$

Which is called the q-analogue of $n \in N$, since

$$\lim_{q\to 1^-}[n]_q = \lim_{q\to 1^-} \sum_{0\le i\le n-1} q^i = n.$$

The Pochhammer symbol $(.)_k$, also called the shifted factorial, defined by

$$(z)_k = \prod_{j=0}^{k-1}(z+j) \ ,k \ge 1 \ ,(z)_0 = 1 \,, (-z)_k = 0, if \ \ z < k \,,$$

which in terms of the Gamma function is given by

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma z} \ , \ k = 0,1,2,3,\ldots, \ z \ne 0,-1,-2,\ldots.$$

And ${}_rF_s$ denoted the ordinary hypergeometric series ([4],[11]) with variable z is defined by

$$ {}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} \Big/ z \right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r)_k}{(b_1,\ldots,b_s)_k} \frac{z^k}{k!} \ , \tag{1.7}$$

being $(a_1,\ldots,a_r)_k = \prod_{i=1}^{r}(a_i)_k$, with $\{a_i\}_{i=1}^{r}$ and $\{b_j\}_{j=1}^{s}$ complex numbers subject to the condition that $b_j \ne -n$ with $n \in N - \{0\}$ for $j == 1,2,3,\ldots,s$.

Here we will give some usual definitions and notations used in q-calculus, i.e. the q-analogues of the usual calculus.

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [5,7]

$$\langle a;q \rangle_n = \begin{cases} 1 & , n = 0 \\ \prod_{m=0}^{n-1} \left(1-q^{a+m}\right) & , n = 1,2,3,\ldots \end{cases} \tag{1.8}$$

The relation to Watson's notation, which is also included in the method, is $\langle a;q \rangle_n = (q^a;q)_n$ where

$$(a;q)_n = \begin{cases} 1 & , n = 0 \\ \prod_{m=0}^{n-1} \left(1-aq^m\right) & , n = 1,2,3,\ldots \end{cases}$$

and $$\langle -n;q\rangle_k = \begin{cases} 0 & k>n \\ \dfrac{\langle 1;q\rangle_n}{\langle 1;q\rangle_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, & k\le n \end{cases}, \tag{1.9}$$

also $\langle a;q\rangle_{n+k} = \langle a;q\rangle_n \langle aq^n;q\rangle_k$ (1.10)

and $\lim\limits_{q\to 1^-} \dfrac{(q^z;q)_k}{(1-q)^k} = (z)_k$.

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_{n-k}}, \ 0\le k\le n, \ \ k,n\in N \tag{1.11}$$

and for complex z is defined by

$$\begin{bmatrix} z \\ k \end{bmatrix}_q = \frac{(q^{-z};q)_k}{(q;q)_k}(-1)^k q^{zk-\binom{k}{2}}; \ k\in N \tag{1.12}$$

Let $\{a_i\}_{i=1}^r$ and $\{b_j\}_{j=1}^s$ complex numbers subject to the condition that $b_j \ne q^{-n}$ with $n\in N\setminus\{0\}$ for $j=1,2,3,\dots,s$.

Then the basic hypergeometric or q-hypergeometric ${}_r\phi_s$ series with variable z is defined by

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} \Big/ q;z\right) = \sum_{k\ge 0} \frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}(-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k},$$

where $(a_1,\dots,a_r;q)_k = \prod\limits_{1\le j\le r}(a_j;q)_k$

In addition, for brevity, let us denote by

$$\left[{}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} \Big/ q;z\right)\right]^n = {}_r\phi_s^{\,n}\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} \Big/ q;z\right), \ n=1,2,3,\dots . \tag{1.13}$$

Analogously to the ordinary hypergeometric ${}_{s+1}F_s$ series, the q-hypergemoetric ${}_{s+1}\phi_s$ series is called k-balanced if $b_1 b_2 \dots b_s = q^k a_1 a_2 \dots a_{s+1}$.

The q-hypergeomwtric ${}_r\phi_s$ series is a q-analogue of the ordinary hypergeometric ${}_rF_s$ series defined by

$$\lim_{q\to 1^-} {}_r\phi_s\left(\begin{matrix} q^{a_1},\dots,q^{a_r} \\ q^{b_1},\dots,q^{b_s} \end{matrix} \Big/ q; z(q-1)^{1+s-r} \right) = {}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} \Big/ z \right).$$

Taking into account the following relations [14,p.18,19]

$${}_2\phi_1\left[\begin{matrix} -n, b \\ c \end{matrix} \Big/ q; q^{n+c-b} \right] = \frac{\langle c-b;q\rangle_n}{\langle c;q\rangle_n}, \quad n = 0,1,2,3,\dots \tag{1.14}$$

where $\langle -n;q\rangle_k = 0$, whenever $n<k$,

and

$$\begin{aligned} \frac{\langle a;q\rangle_n}{\langle 1;q\rangle_n} = {}_2\phi_1\left[\begin{matrix} -n, 1-a \\ 1 \end{matrix} \Big/ q; q^{n+a} \right] &= \sum_{k\ge 0} \frac{\langle -n;q\rangle_k \langle 1-a;q\rangle_k}{\langle 1;q\rangle_k \langle 1;q\rangle_k} q^{k(n+a)} \\ &= \sum_{0\le k\le n} \frac{\langle -n;q\rangle_k \langle 1-a;q\rangle_k}{\langle 1;q\rangle_k \langle 1;q\rangle_k} q^{k(n+a)} \end{aligned} \tag{1.15}$$

$$\begin{aligned} \frac{\langle 1;q\rangle_n}{\langle a+1;q\rangle_n} = {}_2\phi_1\left[\begin{matrix} -n, a \\ a+1 \end{matrix} \Big/ q; q^{n+1} \right] &= \sum_{k\ge 0} \frac{\langle -n;q\rangle_k \langle a;q\rangle_k}{\langle 1;q\rangle_k \langle a+1;q\rangle_k} q^{k(n+1)} \\ &= \sum_{0\le k\le n} \frac{\langle -n;q\rangle_k \langle a;q\rangle_k}{\langle 1;q\rangle_k \langle a+1;q\rangle_k} q^{k(n+1)} \end{aligned} \tag{1.16}$$

In [14] the following two relations are given by

$$\langle a;q\rangle_n = \langle a;q\rangle_{n+1} + q^{n+a} \langle a;q\rangle_n \tag{1.17}$$

$$\langle a;q\rangle_n = \langle 1;q\rangle_n \sum_{k=0}^{n} q^{(a+1)k} \frac{\langle -1;q\rangle_k \langle a+1;q\rangle_{n-k}}{\langle 1;q\rangle_k \langle 1;q\rangle_{n-k}}. \tag{1,18}$$

2. Main Results

In this section we establish some representations for the q-analogue of the multiple Hurwitz Zeta function which defined by

$$\zeta_n(s,w;q)=\sum_{r_1,\ldots,r_n\ge 0}\frac{q^{(w+r_1+\ldots+r_n)(s-1)}}{\{w+r_1+\ldots+r_n\}_q^s},\quad 0<q<1,\quad 0<w\le 1 \quad . \qquad (2.1)$$

where $\{z\}_q$ denotes a q-analogue of a complex number defined by

$$\{z\}_q=\frac{1-q^z}{1-q},\quad q\in C\setminus\{1\}$$

The series (2.1) we can be rewritten as

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{r_1,\ldots,r_n\ge 0}\frac{\langle w;q\rangle_{r_1+\ldots+r_n}^s}{\langle w+1;q\rangle_{r_1+\ldots+r}^s}q^{(r_1+\ldots+r)(s-1)}. \qquad (2.2)$$

Theorem

Let s be an integer number, with $s>1$, $|q|<1$ and $0<w\le 1$. Then the q-analogue of the multiple Hurwitz zeta function (1.3), admits the following representations

i. $$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k=0}^{\infty}(-)^k q^{\binom{k}{2}+wk}\,{}_2\phi_1\left[\begin{matrix}-k,2w\\ w+1\end{matrix}\Big/ q;q^{1-w+k}\right]{}_2\phi_1^{s-1}\left[\begin{matrix}-k,1\\ w+1\end{matrix}\Big/ q;q\right]$$

$${}_{s+1}\phi_s\left[\begin{matrix}k+1,k+1,k+w,\ldots,k+w\\ k+w+1,\ldots,k+w+1\end{matrix}\Big/ q;q^{s-1}\right]{}_{s+2}\phi_{s+1}\left[\begin{matrix}1,k+1+r_1,k+1+r_1,k+w,\ldots,k+w\\ 1+r_1,k+w+1,\ldots,k+w+1\end{matrix}\Big/ q;q^{s-1}\right]$$

$$\ldots{}_{s+2}\phi_{s+1}\left[\begin{matrix}1,k+1+r_1+\ldots+r_{n-1},k+1+r_1+\ldots+r_{n-1},k+w+r_1+\ldots+r_{n-1},\ldots,k+w+r_1+\ldots+r_{n-1}\\ 1+r_1+\ldots+r_{n-1},k+w+1+r_1+\ldots+r_{n-1},\ldots,k+w+1+r_1+\ldots+r_{n-1}\end{matrix}\Big/ q;q^{s-1}\right], \qquad (2.3)$$

ii. $$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k=0}^{\infty}(-)^k q^{\binom{k}{2}+wk}\,{}_2\phi_1\left[\begin{matrix}-k,2w\\ w+1\end{matrix}\Big/ q;q^{1-w+k}\right]{}_2\phi_1\left[\begin{matrix}-k,1-w\\ 1\end{matrix}\Big/ q;q^{w+k}\right]$$

$${}_s\phi_{s-1}\left[\begin{matrix}k+w,\ldots,k+w\\ k+w+1,\ldots,k+w+1\end{matrix}\Big/ q;q^{s-1}\right]{}_{s+1}\phi_s\left[\begin{matrix}1,k+w,\ldots,k+w\\ 1+r_1,k+w+1,\ldots,k+w+1\end{matrix}\Big/ q;q^{s-1}\right]$$

$$\ldots{}_{s+1}\phi_s\left[\begin{matrix}1,k+w+r_1+\ldots+r_{n-1},\ldots,k+w+r_1+\ldots+r_{n-1}\\ 1+r_1+\ldots+r_{n-1},k+w+1+r_1+\ldots+r_{n-1},\ldots,k+w+1+r_1+\ldots+r_{n-1}\end{matrix}\Big/ q;q^{s-1}\right], \qquad (2.4)$$

iii. $\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k=0}^{\infty}q^{(w+1)k}\,{}_2\phi_1\left[\begin{matrix}-k,w+2\\ w+1\end{matrix}\Big/ q;q^{k-1}\right]{}_2\phi_1^{s-1}\left[\begin{matrix}-k,1\\ w+1\end{matrix}\Big/ q;q\right]$

$${}_{s+1}\phi_s\left[\begin{matrix}k+1,w+1,k+w,\ldots,k+w\\ k+w+1,\ldots,k+w+1\end{matrix}\Big/ q;q^{s-1}\right]{}_{s+2}\phi_{s+1}\left[\begin{matrix}1,k+1+r_1,k+1+r_1,k+w+r_1,\ldots,k+w+r_1\\ 1+r_1,k+w+1+r_1,\ldots,k+w+1+r_1\end{matrix}\Big/ q;q^{s-1}\right]$$

$$\ldots{}_{s+2}\phi_{s+1}\left[\begin{matrix}1,k+1+r_1+\ldots+r_{n-1},k+1+r_1+\ldots+r_{n-1},k+w+r_1+\ldots+r_{n-1},\ldots,k+w+r_1+\ldots+r_{n-1}\\ 1+r_1+\ldots+r_{n-1},k+w+1+r_1+\ldots+r_{n-1},\ldots,k+w+1+r_1+\ldots+r_{n-1}\end{matrix}\Big/ q;q^{s-1}\right], \qquad (2.5)$$

iv. $\zeta_n(s,w;q)=(1-q^w)q^{w(s-1)}\{w\}_q^{-s}\,{}_s\phi_{s-1}\left[\begin{matrix}1,w,\ldots,w\\ w+1,\ldots,w+1\end{matrix}\Big/ q;q^s\right]$

$${}_s\phi_{s-1}\left[\begin{matrix}1,w+r_1,\ldots,w+r_1\\ w+1+r_1,\ldots,w+1+r_1\end{matrix}\Big/ q;q^s\right]\ldots{}_s\phi_{s-1}\left[\begin{matrix}1,w+r_1+\ldots+r_{n-1},\ldots,w+r_1+\ldots+r_{n-1}\\ w+1+r_1+\ldots+r_{n-1},\ldots,w+1+r_1+\ldots+r_{n-1}\end{matrix}\Big/ q;q^s\right]$$

$$+q^{ws}\{w\}_q^{-s}\,{}_{s+1}\phi_s\left[\begin{matrix}1,w,\ldots,w\\ w+1,\ldots,w+1\end{matrix}\Big/ q;q^s\right]\ldots{}_{s+1}\phi_s\left[\begin{matrix}1,w+r_1+\ldots+r_{n-1},\ldots,w+r_1+\ldots+r_{n-1}\\ w+1+r+\ldots+r_{n-11},\ldots,w+1+r_1+\ldots+r_{n-1}\end{matrix}\Big/ q;q^s\right]. \qquad (2.6)$$

Proof.

From (2.2) and using relation (1.15), we get

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{r_1,\ldots,r_n\ge 0}\frac{\langle w;q\rangle_{r_1+\ldots+r_n}^{s-1}}{\langle w+1;q\rangle_{r_1+\ldots+r}^{s}}\langle 1;q\rangle_{r_1+\ldots+r_n}q^{(r_1+\ldots+r_n)(s-1)}$$
$$\times\sum_{0\le k\le r_1,\ldots,r_n}\frac{\langle-(r_1+\ldots+r_n);q\rangle_k\langle 1-w;q\rangle_k}{\langle 1;q\rangle_k\langle 1;q\rangle_k}q^{k((r_1+\ldots+r_n)+w)}$$

Then, using (1,9) with replacing n by $r_1+r_2+\ldots+r_n$, we get

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k\ge 0}\sum_{r_1,\ldots,r_n\ge 0}\frac{\langle w;q\rangle_{r_1+\ldots+r_n+k}^{s-1}\langle 1-w;q\rangle_k\langle 1;q\rangle_{r_1+\ldots+r_n+k}\langle 1;q\rangle_{r_1+\ldots+r_n+k}}{\langle w+1;q\rangle_{r_1+\ldots+r_n+k}^{s}\langle 1;q\rangle_k\langle 1;q\rangle_k\langle 1;q\rangle_{r_1+\ldots+r_n}}$$
$$\times(-q^w)^k q^{\binom{k}{2}+(r_1+\ldots+r_n+k)(s-1)}$$

$$=q^{w(s-1)}\{w\}_q^{-s}\sum_{k\ge 0}(-1)^k q^{\binom{k}{2}+wk}\frac{\langle 1-w;q\rangle_k}{\langle w+1;q\rangle_k}\frac{\langle w;q\rangle_k^{s-1}}{\langle w+1;q\rangle_k^{s-1}}q^{k(s-1)}$$

$$\times\sum_{r_1,\ldots,r_n\ge 0}\frac{\langle 1+k;q\rangle_{r_1+\ldots+r_n}\langle 1+k;q\rangle_{r_1+\ldots+r_n}\langle k+w;q\rangle_{r_1+\ldots+r_n}^{s-1}}{\langle 1;q\rangle_{r_1+\ldots+r_n}\langle k+w+1;q\rangle_{r_1+\ldots+r_n}^{s}}q^{(r_1+\ldots+r_n)(s-1)}$$

By using relation (1.14), we obtain

$$=q^{w(s-1)}\{w\}_q^{-s}\sum_{k\geq 0}(-1)^k q^{\binom{k}{2}+wk}\,{}_2\phi_1\left[\begin{matrix}-k,2w\\ w+1\end{matrix}\Big/ q;q^{1-w+k}\right]{}_2\phi_1^{s-1}\left[\begin{matrix}-k,1\\ w+1\end{matrix}\Big/ q;q\right]$$

$$\times\sum_{r_1\geq 0}\frac{\langle 1+k;q\rangle_{r_1}\langle 1+k;q\rangle_{r_1}\langle k+w;q\rangle_{r_1}^{s-1}}{\langle 1;q\rangle_{r_1}\langle k+w+1;q\rangle_{r_1}^{s}}q^{r_1(s-1)}$$

$$\times\sum_{r_2\geq 0}\frac{\langle 1+k+r_1;q\rangle_{r_2}\langle 1+k+r_1;q\rangle_{r_2}\langle k+w+r_1;q\rangle_{r_2}^{s-1}}{\langle 1+r_1;q\rangle_{r_2}\langle k+w+1+r_1;q\rangle_{r_2}^{s}}q^{r_2(s-1)}$$

$$\vdots$$

$$\times\sum_{r_2\geq 0}\frac{\langle 1+k+r_1+\ldots+r_{n-1};q\rangle_{r_n}\langle 1+k+r_1+\ldots+r_{n-1};q\rangle_{r_n}\langle k+w+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s-1}}{\langle 1+r_1+\ldots+r_{n-1};q\rangle_{r_n}\langle k+w+1+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s}}q^{r_n(s-1)}$$

which is required (2.3).

Similarly, From (2.2) and the relation (1.16) with replacing n by $r_1+r_2+\ldots+r_n$ we obtain

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{r_1,\ldots,r_n\geq 0}\frac{\langle w;q\rangle_{r_1+\ldots+r_n}^{s}}{\langle w+1;q\rangle_{r_1+\ldots+r_n}^{s-1}}\frac{q^{(r_1+\ldots+r_n)(s-1)}}{\langle 1;q\rangle_{r_1+\ldots+r_n}}$$

$$\times\sum_{0\leq k\leq r_1,\ldots,r_n}\frac{\langle -(r_1+\ldots+r_n);q\rangle_k\langle w;q\rangle_k}{\langle 1;q\rangle_k\langle w+1;q\rangle_k}q^{k(r_1+\ldots+r_n+1)}$$

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k\geq 0}(-1)^k q^{\binom{k+1}{2}}\frac{\langle w;q\rangle_k}{\langle w+1;q\rangle_k}\frac{\langle w;q\rangle_k}{\langle 1;q\rangle_k}\frac{\langle w;q\rangle_k^{s-1}}{\langle w+1;q\rangle_k^{s-1}}q^{k(s-1)}$$

$$\times\sum_{r_1,\ldots,r_n\geq 0}\frac{\langle k+w;q\rangle_{r_1+\ldots+r_n}^{s}}{\langle 1;q\rangle_{r_1+\ldots+r_n}\langle k+w+1;q\rangle_{r_1+\ldots+r_n}^{s-1}}q^{(r_1+\ldots+r_n+k)(s-1)}$$

Which by using relation (1.14), we find

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}$$
$$\times\sum_{k\geq 0}(-1)^k q^{\binom{k+1}{2}}\,{}_2\phi_1\left[\begin{matrix}-k,1\\ w+1\end{matrix}\Big/ q;q^{k+w}\right]{}_2\phi_1\left[\begin{matrix}-k,1-w\\ 1\end{matrix}\Big/ q;q^{k+w}\right]{}_2\phi_1^{s-1}\left[\begin{matrix}-k,1\\ w+1\end{matrix}\Big/ q;q\right]$$
$$\times\sum_{r_1\geq 0}\frac{\langle k+w;q\rangle_{r_1}^{s}}{\langle 1;q\rangle_{r_1}\langle k+w+1;q\rangle_{r_1}^{s-1}}q^{r_1(s-1)}\sum_{r_2\geq 0}\frac{\langle k+w+r_1;q\rangle_{r_2}^{s}}{\langle 1+r_1;q\rangle_{r_1}\langle k+w+1+r_1;q\rangle_{r_2}^{s-1}}q^{r_2(s-1)}$$
$$\vdots$$
$$\times\sum_{r_n\geq 0}\frac{\langle k+w+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s}}{\langle 1+r_1+\ldots+r_{n-1};q\rangle_{r_n}\langle k+w+1+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s-1}}q^{r_n(s-1)}$$

which is required (2.4).

Now, by using expression (1.18) in (2.2), we get

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k\geq 0}\sum_{r_1,\ldots,r_n\geq 0}\frac{\langle w;q\rangle_{r_1+\ldots+r_n+k}^{s-1}\langle 1;q\rangle_{r_1+\ldots+r_n+k}}{\langle w+1;q\rangle_{r_1+\ldots+r_n+k}^{s}\langle 1;q\rangle_{k}}$$
$$\times(q)^{(w+1)k}\frac{\langle -1;q\rangle_k\langle w+1;q\rangle_{r_1+\ldots+r_n}}{\langle 1;q\rangle_{r_1+\ldots+r_n}}q^{(r_1+\ldots+r_n+k)(s-1)}$$

By using expression (1.10), we find

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k\geq 0}(q)^{(w+1)k}\frac{\langle -1;q\rangle_k}{\langle w+1;q\rangle_k}\frac{\langle 1;q\rangle_k}{\langle 1;q\rangle_k}\frac{\langle w;q\rangle_k^{s-1}}{\langle w+1;q\rangle_k^{s-1}}q^{k(s-1)}$$
$$\times\sum_{r_1,\ldots,r_n\geq 0}\frac{\langle 1+k;q\rangle_{r_1+\ldots+r_n}\langle w+1;q\rangle_{r_1+\ldots+r_n}\langle w+k;q\rangle_{r_1+\ldots+r_n}^{s-1}}{\langle 1;q\rangle_{r_1+\ldots+r_n}\langle w+1+k;q\rangle_{r_1+\ldots+r_n}^{s}}q^{(r_1+\ldots+r_n)(s-1)}$$

By using relation (1.14), we obtain

$$\zeta_n(s,w;q)=q^{w(s-1)}\{w\}_q^{-s}\sum_{k\geq 0}(q)^{(w+1)k}\,{}_2\phi_1\left[\begin{matrix}-k,w+2\\ w+1\end{matrix}\Big/ q;q^{k-1}\right]{}_2\phi_1^{s-1}\left[\begin{matrix}-k,1\\ w+1\end{matrix}\Big/ q;q\right]$$
$$\times\sum_{r_1\geq 0}\frac{\langle 1+k;q\rangle_{r_1}\langle w+1;q\rangle_{r_1}\langle w+k;q\rangle_{r_1}^{s-1}}{\langle 1;q\rangle_{r_1}\langle w+1+k;q\rangle_{r_1}^{s}}q^{r_1(s-1)}$$

$$\times\sum_{r_2\geq 0}\frac{\langle 1+k+r_1;q\rangle_{r_2}\langle w+1+r_1;q\rangle_{r_2}\langle w+k+r_1;q\rangle_{r_2}^{s-1}}{\langle 1+r_1;q\rangle_{r_2}\langle w+1+k+r_1;q\rangle_{r_2}^{s}}q^{(r_2)(s-1)}$$
$$\vdots$$

$$\times \sum_{r_n \geq 0} \frac{\langle 1+k+r_1+\cdots+r_{n-1};q\rangle_{r_n} \langle w+1+r_1+\cdots+r_{n-1};q\rangle_{r_n} \langle w+k+r_1+\cdots+r_{n-1};q\rangle_{r_n}^{s-1}}{\langle 1+r_1+\cdots+r_{n-1};q\rangle_{r_n} \langle w+1+k+r_1+\cdots+r_{n-1};q\rangle_{r_n}^{s}} q^{(r_n)(s-1)}$$

which is required (2.5).

Finally, using relation (1.17) in (2.2), we get

$$\zeta_n(s,w;q) = q^{w(s-1)}\{w\}_q^{-s} \sum_{r_1,\ldots,r_n \geq 0} \frac{\langle w;q\rangle_{r_1+\ldots+r_n}^{s-1} \langle w;q\rangle_{r_1+\ldots+r_n+1}}{\langle w+1;q\rangle_{r_1+\ldots+r_n}^{s-1} \langle w+1;q\rangle_{r_1+\ldots+r_n}} q^{(r_1+\ldots+r_n)(s-1)}$$

$$+ q^{ws}\{w\}_q^{-s} \sum_{r_1,\ldots,r_n \geq 0} \frac{\langle w;q\rangle_{r_1+\ldots+r_n}^{s}}{\langle w+1;q\rangle_{r_1+\ldots+r_n}^{s}} q^{(r_1+\ldots+r_n)s}$$

since $\langle w+1;q\rangle_{r_1+\ldots+r_n} = \dfrac{1-q^{r_1+\ldots+r_n+w}}{1-q^w}\langle w;q\rangle_{r_1+\ldots+r_n}$

and $\langle w+1;q\rangle_{r_1+\ldots+r_n+1} = \langle 1-q^{r_1+\ldots+r_n+w}\rangle \langle w;q\rangle_{r_1+\ldots+r_n}$

we deduce

$$\zeta_n(s,w;q) = (1-q^w) q^{w(s-1)}\{w\}_q^{-s} \sum_{r_1,\ldots,r_n \geq 0} \frac{\langle w;q\rangle_{r_1+\ldots+r_n}^{s-1}}{\langle w+1;q\rangle_{r_1+\ldots+r_n}^{s-1}} q^{(r_1+\ldots+r_n)(s-1)}$$

$$+ q^{ws}\{w\}_q^{-s} \sum_{r_1,\ldots,r_n \geq 0} \frac{\langle w;q\rangle_{r_1+\ldots+r_n}^{s}}{\langle w+1;q\rangle_{r_1+\ldots+r_n}^{s}} q^{(r_1+\ldots+r_n)s}$$

By using relation (1.10), we get

$$\zeta_n(s,w;q) = (1-q^w) q^{w(s-1)}\{w\}_q^{-s} \sum_{r_1 \geq 0} \frac{\langle w;q\rangle_{r_1}^{s-1}}{\langle w+1;q\rangle_{r_1}^{s-1}} q^{(r_1)(s-1)} \sum_{r_2 \geq 0} \frac{\langle w+r_1;q\rangle_{r_2}^{s-1}}{\langle w+1+r_1;q\rangle_{r_2}^{s-1}} q^{(r_2)(s-1)}$$

$$\ldots \sum_{r_n \geq 0} \frac{\langle w+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s-1}}{\langle w+1+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s-1}} q^{(r_n)(s-1)}$$

$$+ q^{ws}\{w\}_q^{-s} \sum_{r_1 \geq 0} \frac{\langle w;q\rangle_{r_1}^{s}}{\langle w+1;q\rangle_{r_1}^{s}} q^{(r_1)(s)} \sum_{r_2 \geq 0} \frac{\langle w+r_1;q\rangle_{r_2}^{s}}{\langle w+1+r_1;q\rangle_{r_2}^{s}} q^{(r_2)(s)}$$

$$\ldots \sum_{r_n \geq 0} \frac{\langle w+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s}}{\langle w+1+r_1+\ldots+r_{n-1};q\rangle_{r_n}^{s}} q^{(r_n)(s)}$$

which is required (2.6).

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