



## Quasi 2-metrics, 2-metrics and their decomposition

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### Abstract

In this paper, we define the quasi semi 2-metric and quasi 2-metric functions, and after that we give some ways to decompose a 2-metric in two conjugated quasi 2-metrics.

**Keywords:** Quasi semi 2-metric; quasi 2-metric; decomposition.

### 1. Introduction

It is well known that, if one attempts to omit the requirement of symmetry in a semi metric or semi norm, then the structure of the space is drastically changed. Such unsymmetrical spaces have been studied by Wilson [1] who used the term quasi semi metric when the feature of symmetry is omitted in a semi metric. Many classical results of the analysis have been extended to such non-symmetric spaces like in [2], [3],[4].

In [5] J.M. Hernandez and C.H. Castaneda gave some ways to decompose metrics and norms. In this paper we will try to do the same for the 2-metric function.

Let us first give the definitions for the quasi semi 2-metric and quasi 2-metric.

#### Definition 1.1

A quasi semi 2-metric function is a mapping:  $\sigma: X \times X \times X \rightarrow \mathbb{R}^+$  such that:

(QSM1)  $\forall x, y \in X, \exists z \in X$ , so that  $\sigma(x, y, z) \neq 0$ ,

(QSM2)  $\sigma(x, y, z) = 0$  if at least two of three elements are equal.

If further (QSM3):  $\sigma(x, y, z) = \sigma(x, z, y) = \sigma(z, y, x)$  holds, then  $\sigma$  is the semi 2-metric function.

#### Definition 1.2

A 2-metric is a mapping  $\sigma: X \times X \times X \rightarrow \mathbb{R}^+$  such that:

- 1)  $\sigma(x, y, z) \neq 0$
- 2)  $\sigma(x, y, z) = 0$  when two of the three elements are equal
- 3)  $\sigma(x, y, z) \leq \sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z)$
- 4)  $\sigma(x, y, z) = \sigma(x, z, y) = \sigma(y, z, x)$  for every  $x, y, z, u \in X$ .

If we try to omit the requirement of the symmetry in definition 1.2 (the 4-rth property) we will obtain the concept of the quasi 2-metric:

#### Definition 1.3

A quasi 2-metric is a mapping  $\rho: X \times X \times X \rightarrow \mathbb{R}^+$  such that the following properties hold:

- 1)  $\rho(x, y, z) \neq 0$
- 2)  $\rho(x, y, z) = 0$  when two of the three elements are equal

$$3) \rho(x, y, z) \leq \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z).$$

The pair  $(X, \rho)$  is called the quasi 2-metric space.

**Example 1.4**  $\mathbb{R}$  is a quasi 2-metric space, with  $\rho(x, y, z) = \begin{cases} xy & \text{if } xy \geq 0 \\ -\frac{xy}{2} & \text{if } xy < 0 \end{cases}$ .

A quasi semi 2-metric  $\sigma$  on  $X$ , has its conjugate  $\bar{\sigma}$  defined as:

$$\bar{\sigma}(x, y, z) = \sigma(x, y, z) \text{ for every } x, y, z \in X.$$

It is easy to show that  $\bar{\sigma}(x, y, z)$  is a quasi semi 2-metric. Mean while the function  $\sigma^s: X \times X \rightarrow \mathbb{R}$  such that:

$$\sigma^s(x, y, z) = \max \{ \sigma(x, y, z), \bar{\sigma}(x, y, z) \} \text{ is a semi 2-metric.}$$

### Definition 1.5

If  $\sigma$  is a 2-metric on  $X$  and for some quasi 2-metric  $\rho \neq \sigma$  is satisfied that  $\rho^s = \sigma$  then the pair  $(\rho, \bar{\rho})$  is called the *decomposition of the 2-metric  $\sigma$* .

## 2. Main results

### Theorem 2.1

Let  $(X, \sigma)$  be a 2-metric space. If  $f: X \rightarrow \mathbb{R}$  is a function satisfying:  $\sigma(u, y, z) \geq \sigma(x, y, z)$  and  $\sigma(x, y, u) \geq \sigma(x, y, z)$  whenever  $f(x) \leq f(y) \leq f(u)$ , then there exists a decomposition of  $\sigma$ .

**Proof.**

$$\text{Let: } \rho(x, y, z) = \rho_f(x, y, z) = \begin{cases} \sigma(x, y, z) & \text{if } f(x) \leq f(y) \\ 0 & \text{if } f(x) > f(y) \end{cases}.$$

Let us prove that  $\rho$  is a quasi 2-metric, we will start showing the triangle inequality by cases, for  $x, y, z, u \in X$ .

Case 1: If  $f(x) > f(y)$

$$\rho(x, y, z) = 0 \leq \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z).$$

Case 2: If  $f(x) \leq f(y)$ . There are 3 subcases:

Subcase 1: If  $f(x) \leq f(y) < f(u)$

$$\rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z) = \sigma(x, y, u) \geq \sigma(x, y, z) = \rho(x, y, z).$$

Subcase 2: If  $f(x) \leq f(u) \leq f(y)$

$$\begin{aligned} \rho(x, y, z) &= \sigma(x, y, z) \leq \sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z) \\ &= \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z). \end{aligned}$$

Subcase 3: If  $f(u) < f(x) \leq f(y)$

$$\rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z) = \sigma(u, y, z) \geq \sigma(x, y, z) = \rho(x, y, z).$$

In addition it is clear that  $\rho(x, y, z) \geq 0$  and if  $\rho(x, y, u) = \rho(x, u, z) = \rho(u, y, z) = 0$  then  $\sigma(x, y, z) = 0$  that means:  $x = y$  or  $y = z$  or  $x = z$ .

Furthermore,  $\bar{\rho}(x, y, z) = \rho(y, x, z) = \begin{cases} \sigma(y, x, z) & \text{if } f(y) \leq f(x) \\ 0 & \text{if } f(y) > f(x) \end{cases}$ , thus:

$$\rho^s(x, y, z) = \max \{ \rho(x, y, z), \bar{\rho}(x, y, z) \} = \sigma(x, y, z). \blacksquare$$

### Theorem 2.2

Let  $(X, \sigma)$  be a 2-metric space. If  $\text{card}(X) \leq \aleph_2$  then there exists a decomposition of  $\sigma$ .

**Proof.** If  $\text{card}(X) \leq \aleph_2$  then there exist  $f: X \rightarrow \mathbb{R}$  an injective function. Given  $k > 1$  we define:

$$\rho_k(x, y, z) = \rho_{f,k}(x, y, z) = \begin{cases} \sigma(x, y, z) & \text{if } f(x) \geq f(y) \\ \frac{\sigma(x, y, z)}{k} & \text{if } f(x) < f(y) \end{cases}.$$

It is clear that if:  $\rho_k(x, y, z) = \rho_k(y, x, z) = 0 \Rightarrow \sigma(x, y, z) = \frac{\sigma(y, x, z)}{k} = 0$

or:  $\sigma(y, x, z) = \frac{\sigma(x, y, z)}{k} = 0 \Rightarrow x = y$ .

For  $x, y, z, u \in X$  we have the following cases:

Case 1: If  $f(x) \geq f(y) \geq f(u)$

$$\begin{aligned} \rho_k(x, y, z) &= \sigma(x, y, z) \leq \sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z) \\ &= \rho(x, y, u) + \rho(x, u, z) + k\rho(u, y, z) \\ &\leq k(\rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z)). \end{aligned}$$

Case 2: If  $f(x) > f(u) > f(y)$

$$\begin{aligned} \rho_k(x, y, z) &= \sigma(x, y, z) \leq \sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z) \\ &= \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z). \end{aligned}$$

Case 3: If  $f(u) \geq f(x) \geq f(y)$

$$\begin{aligned} \rho_k(x, y, z) &= \sigma(x, y, z) \leq \sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z) \\ &= \rho(x, y, u) + k\rho(x, u, z) + \rho(u, y, z) \\ &\leq k(\rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z)). \end{aligned}$$

Case 4: If  $f(x) < f(y) \leq f(u)$

$$\begin{aligned} \rho_k(x, y, z) &= \frac{\sigma(x, y, z)}{k} \leq \frac{1}{k} [\sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z)] \\ &= \frac{1}{k} [k\rho(x, y, u) + k\rho(x, u, z) + \rho(u, y, z)] \\ &= \rho(x, y, u) + \rho(x, u, z) + \frac{1}{k}\rho(u, y, z) \\ &\leq \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z). \end{aligned}$$

Case 5: If  $f(x) < f(u) < f(y)$

$$\begin{aligned} \rho_k(x, y, z) &= \frac{\sigma(x, y, z)}{k} \leq \frac{1}{k} [\sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z)] \\ &= \frac{1}{k} [k\rho(x, y, u) + k\rho(x, u, z) + k\rho(u, y, z)] \\ &= \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z). \end{aligned}$$

Case 6: If  $f(u) \leq f(x) < f(y)$

$$\begin{aligned} \rho_k(x, y, z) &= \frac{\sigma(x, y, z)}{k} \leq \frac{1}{k} [\sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z)] \\ &= \frac{1}{k} [k\rho(x, y, u) + \rho(x, u, z) + k\rho(u, y, z)] \\ &= \rho(x, y, u) + \frac{1}{k}\rho(x, u, z) + \rho(u, y, z) \\ &\leq \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z). \end{aligned}$$

Because of the definition of  $\rho_k^s$  it is easy to show that  $\rho_k^s = \sigma$ . ■

### Theorem 2.3

Let  $(X, \sigma)$  be a 2-metric space with  $card(X) \leq \aleph_2$ , let  $k, l > 1$  and  $f, g: X \rightarrow \mathbb{R}$  be injective functions. Then  $\rho_{f,k} \equiv \rho_{g,l}$  (2-metrically).

**Proof.** Let  $x, y, z \in X$ , we may assume that  $k < l$ . If  $\rho_{f,k}$  and  $\rho_{g,l}$  are given by:

$$\rho_{f,k} = \begin{cases} \sigma(x, y, z) & \text{if } f(x) - f(y) \geq 0 \\ \frac{\sigma(x, y, z)}{k} & \text{if } f(x) - f(y) < 0 \end{cases}$$

and:

$$\rho_{g,l} = \begin{cases} \sigma(x, y, z) & \text{if } g(x) - g(y) \geq 0 \\ \frac{\sigma(x, y, z)}{l} & \text{if } g(x) - g(y) < 0 \end{cases}$$

Case 1: If  $f(x) - f(y) \geq 0$  and  $g(x) - g(y) \geq 0$ , then  $\rho_{f,k} = \sigma(x, y, z) = \rho_{g,l}$ .

Case 2: If  $f(x) - f(y) < 0$  and  $g(x) - g(y) < 0$ , then:

$$\rho_{f,k} = \frac{\sigma(x, y, z)}{k} > \frac{\sigma(x, y, z)}{l} = \rho_{g,l}.$$

Case 3: If  $f(x) - f(y) \geq 0$  and  $g(x) - g(y) < 0$ , then:

$$\rho_{f,k} = \sigma(x, y, z) > \frac{\sigma(x, y, z)}{l} = \rho_{g,l}.$$

Case 4: If  $f(x) - f(y) < 0$  and  $g(x) - g(y) \geq 0$ , then:

$$\rho_{f,k} = \frac{\sigma(x, y, z)}{k} > \sigma(x, y, z) = \rho_{g,l}.$$

And finally, if we perform a similar analysis to the Theorem 2.3, we obtain that for all  $x, y, z \in X$ :

$$k\rho_{f,k}(x, y, z) \geq \rho_{g,l}(x, y, z) \text{ and } l\rho_{g,l}(x, y, z) \geq \rho_{f,k}(x, y, z).$$

Hence:

$$k\rho_{f,k}(x, y, z) \geq \rho_{g,l}(x, y, z) \geq \frac{1}{l}\rho_{f,k}(x, y, z).$$

Therefore:  $\rho_{f,k} \equiv \rho_{g,l}$ . ■

## References

- [1] Wilson. W.A, *On quasi metric spaces*, American J.Math. 53 (1931) 675-84.
- [2] S.Cobzas. *Functional on Asymmetric Normed Spaces*. Springer Fronties in Mathematics, 2013. ISBN. 978-3-0348-0478-3.
- [3] Hernandez. J.M, C.H.Castaneda, L.C. Alvarez, Hernandez. J.L, V.Tochihuitl, J.L. Carrasco, Ramirez T.M and R.Vazquez, *Espacios con distancias no simetricas*, Temas de Ciencia y Tecnologia, Vol.18, No.53, (2014), 3-9.
- [4] Jimenez-Pozo.M.A. and J.M.Hernandez, *Asymmetric Holder spaces of sign sensitive weighted integrable functions*, Comn.Math.App, Vol.3 (2011), 39-50.
- [5] J.M.Hernandez, C.H.Castaneda, L.C. Alvarez, V.Tochihuitl and R.Vazquez, *Decomposition of metrics and norms*, GJMMS, ISSN 0972-9836, Vol.6, Nr. 1 (2016), pp.1-9.