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Decomposition of 2-norms

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Abstract.

In this paper, after introducing the notion of asymmetric semi 2-norm and asymmetric 2-norm, we present the conjugate function of an asymmetric 2-norm. Then we give a decomposition of a 2-norm in two conjugated asymmetric 2-norms.

Keywords: Asymmetric semi 2-norm; conjugate function of an asymmetric 2-norm; decomposition of a 2-norm.

1. Introduction

In 1968 Duffin and Karlovitz [1] proposed the term asymmetric semi norm when the property of symmetry is not satisfied in a semi norm. Many classical results of the analysis have been extended to such non-symmetric spaces as in [2],[3].

Let first give the definition of the asymmetric 2-norm function.

Definition1.1. Let X be an vectorial space. The function $\|.,.\|: X^2 \to R^+$ it is called the asymmetric 2-norm function if:

1) $\|x,x\| \geq 0$ for $\forall x \in X$ and $\|x,x\| = 0 \Leftrightarrow x = 0$
2) $\|x,y\| = \|y,x\|$ for $x,y \in X$
3) $\|\lambda x, y\| = \lambda \|x,y\|$ for $\lambda > 0$ and $x,y \in X$
4) $\|x+y,z\| \leq \|x,z\| + \|y,z\|$ for $x,y,z \in X$.

The pair $(X, \|.,.\|)$ is then called an asymmetric 2-normed space.

Example1.2. Let $X = R$ and $p(x): R \to R^+$ an asymmetric norm on R, (defined in [4]). For every two points $x(x_1,x_2)$ and $y(y_1,y_2)$ on R^2 we remark: $\|x,y\| = p(x_1)p(y_1) + p(x_2)p(y_2)$.

Claim1.3. The function $\|x,y\| = p(x_1)p(y_1) + p(x_2)p(y_2)$ is an asymmetric 2-norm function on R .

Proof:

1) $\|x,x\| \geq 0$ for $\forall x \in X$ and $\|x,x\| = 0 \Leftrightarrow x = 0$

$$\|x,x\| = p(x_1)p(x_1) + p(x_2)p(x_2) = p^2(x_1) + p^2(x_2) > 0$$

$$\|x,x\| = 0 \Leftrightarrow p^2(x_1) + p^2(x_2) = 0 \Leftrightarrow x_1 = x_2 = 0 \text{ for } \forall x \in X .$$

2) $\|x,y\| = \|y,x\|$ for $x,y \in X$

$$\|x,y\| = p(x_1)p(y_1) + p(x_2)p(y_2) = p(y_1)p(x_1) + p(y_2)p(x_2) = \|y,x\|$$

3) $\|\lambda x, y\| = \lambda \|x, y\|$ for $\lambda > 0$ and $x, y \in X$

$\|\lambda x, y\| = p(\lambda x_1)p(y_1) + p(\lambda x_2)p(y_2) = \lambda[p(x_1)p(y_1) + p(x_2)p(y_2)] = \lambda\|x, y\|$

4) $\|x + y, z\| \le \|x, z\| + \|y, z\|$ for $x, y, z \in X$

$$\|x+y,z\| = p(x_1+y_1)p(z_1) + p(x_2+y_2)p(z_2) \le [p(x_1)+p(y_1)]p(z_1) + [p(x_2)+p(y_2)]p(z_2) =$$
$$= p(x_1)p(z_1) + p(x_2)p(z_2) + p(y_1)p(z_1) + p(y_2)p(z_2) = \|x,z\| + \|y,z\|.\blacksquare$$

Definition 1.4. The function $\sigma : X^2 \to R^+$ is called an asymmetric semi 2-norm if:

1) $\sigma(x, y) = 0$ if x, y linearly dependent

2) $\sigma(x, y) = \sigma(y, x)$ for every $x, y \in X$

3) $\sigma(\lambda x, y) = \lambda\sigma(x, y)$, for $\lambda > 0$

4) $\sigma(x, y) = \sigma(x, y - x)$ for every $x, y \in X$.

The analogue definition of the semi 2-norm and 2-norm function is given in [4].

2. Main results

Now the conjugate of the asymmetric semi 2-norm function is defined by:

$$\overline{\sigma}(x, y) = \sigma(-x, -y),\ x, y \in X .$$

The function $\sigma^s : X^2 \to R^+$ given by:

$$\sigma^s(x, y) = \max\left\{\sigma(x, y), \overline{\sigma}(x, y)\right\},\ x, y \in X$$

is an asymmetric semi 2-norm [5].

An asymmetric semi 2-norm σ on X^2 is an asymmetric 2-norm if only if σ^s is a 2-norm on X.

Definition 2.1. If $\|.,.\|$ is a 2-norm on and $\sigma \ne \|.,.\|$ is an asymmetric 2-norm such that $\sigma^s = \|.,.\|$ then the pair (σ, σ^s) is called a decomposition of the 2-norm $\|.,.\|$.

Theorem 2.2. Let $\left(X^2, \|.,.\|\right)$ be an 2-normed space. For each $f : X \to R$ linear function and each $k \ge 1$ there exist a decomposition of $\|.,.\|$.

Proof. Given such f, k, we define $\sigma_{f,k}(x, y) = \begin{cases} \|x, y\|, & f(x) \ge 0 \\ \dfrac{\|x, y\|}{k}, & f(x) < 0 \end{cases}$.

Let's prove $\sigma_{f,k}(x, y)$ is an asymmetric 2-norm.

1) It is clear that: $\sigma_{f,k}(x, x) = \begin{cases} \|x, x\|, & f(x) \ge 0 \\ \dfrac{\|x, x\|}{k}, & f(x) < 0 \end{cases}$. So $\sigma_{f,k}(x, x) \ge 0$ and $\sigma_{f,k}(x, x) = 0 \Leftrightarrow x = 0$.

2) $\sigma_{f,k}(x,y)=\begin{cases}\|x,y\|, & f(x)\geq 0\\ \dfrac{\|x,y\|}{k}, & f(x)<0\end{cases}=\begin{cases}\|y,x\|, & f(x)\geq 0\\ \dfrac{\|y,x\|}{k}, & f(x)<0\end{cases}=\sigma_{f,k}(y,x)$.

3) For $\lambda>0$ we have:

$$\sigma_{f,k}(\lambda x,y)=\begin{cases}\|\lambda x,y\|, & f(x)\geq 0\\ \dfrac{\|\lambda x,y\|}{k}, & f(x)<0\end{cases}=\begin{cases}\lambda\|x,y\|, & f(x)\geq 0\\ \dfrac{\lambda\|x,y\|}{k}, & f(x)<0\end{cases}=\lambda\sigma_{f,k}(x,y).$$

4) For the triangle inequality we have 4 cases for $x,y,z\in X$.

Case 1. If $f(x)\geq 0$ and $f(y)\geq 0$ then:

$$\sigma_{f,k}(x+y,z)=\|x+y,z\|\leq\|x,z\|+\|y,z\|=\sigma_{f,k}(x,z)+\sigma_{f,k}(y,z).$$

Case 2. If $f(x)<0$ and $f(y)<0$ then:

$$\sigma_{f,k}(x+y,z)=\frac{\|x+y,z\|}{k}\leq\frac{\|x,z\|+\|y,z\|}{k}=\frac{\|x,z\|}{k}+\frac{\|y,z\|}{k}=\sigma_{f,k}(x,z)+\sigma_{f,k}(y,z).$$

Case 3. If $f(x)\geq 0$ and $f(y)<0$, in this case we have two possibilities:

1) $f(x+y)\geq 0$, then:

$$\sigma_{f,k}(x+y,z)=\|x+y,z\|\leq\|x,z\|+\|y,z\|=\sigma_{f,k}(x,z)+k\sigma_{f,k}(y,z)\leq$$
$$\leq k\left(\sigma_{f,k}(x,z)+\sigma_{f,k}(y,z)\right)=\sigma_{f,k}(x,z)+\sigma_{f,k}(y,z)\ \text{ for } k=1.$$

2) $f(x+y)<0$, then:

$$\sigma_{f,k}(x+y,z)=\frac{\|x+y,z\|}{k}\leq\frac{\|x,z\|+\|y,z\|}{k}\leq\|x,z\|+\frac{\|y,z\|}{k}=\sigma_{f,k}(x,z)+\sigma_{f,k}(y,z).$$

Case 4. If $f(x)<0$ and $f(y)\geq 0$ is treated similarly to case 3.

Now it remains to prove that $\sigma^s_{f,k}=\|.,.\|$. In order to do this let us identify $\overline{\sigma}_{f,k}$:

$$\overline{\sigma}_{f,k}(x,y)=\sigma_{f,k}(-x,-y)=\begin{cases}\|-x,-y\|, & f(-x)\geq 0\\ \dfrac{\|-x,-y\|}{k}, & f(-x)<0\end{cases}=\begin{cases}\|x,y\|, & f(x)\leq 0\\ \dfrac{\|x,y\|}{k}, & f(x)>0\end{cases}$$

therefore: $\sigma^s_{f,k}=\max\left\{\sigma_{f,k}(x,y),\overline{\sigma}_{f,k}(x,y)\right\}=\|x,y\|$.■

Theorem 2.3. Let $\left(X^2,\|.,.\|\right)$ be an 2-normed space and f,g be functions as in Theorem 2.2 and $k,l\geq 1$. Then: $\sigma_{f,k}\equiv\sigma_{g,l}$.

Proof. Let $x, y \in X$ and assume $k < l$. By Theorem 2.2 there exist $\sigma_{f,k}$ and $\sigma_{g,l}$.

Case 1. If $f(x) \geq 0$ and $g(x) \geq 0$, then:

$$\sigma_{f,k}(x, y) = \|x, y\| = \sigma_{g,l}(x, y).$$

Case 2. If $f(x) < 0$ and $g(x) < 0$, then:

$$\sigma_{f,k}(x, y) = \frac{\|x, y\|}{k} > \frac{\|x, y\|}{l} = \sigma_{g,l}(x, y).$$

Case 3. If $f(x) \geq 0$ and $g(x) < 0$, then:

$$\sigma_{f,k}(x, y) = \|x, y\| > \frac{\|x, y\|}{l} = \sigma_{g,l}(x, y).$$

Case 4. If $f(x) < 0$ and $g(x) \geq 0$, then:

$$\sigma_{f,k}(x, y) = \frac{\|x, y\|}{k} < \|x, y\| = \sigma_{g,l}(x, y).$$

So $k\sigma_{f,k}(x, y)$ satisfies for cases 1,2 and 3 that $k\sigma_{f,k}(x, y) \geq \sigma_{g,l}(x, y)$, and case 4 that $k\sigma_{f,k}(x, y) = \sigma_{g,l}(x, y)$. Likewise $l\sigma_{g,l}(x, y)$ satisfies for cases 1,2 and 4 that $l\sigma_{g,l}(x, y) \geq \sigma_{f,k}(x, y)$ and case 3 that $l\sigma_{g,l}(x, y) = \sigma_{f,k}(x, y)$, i.e. $k\sigma_{f,k}(x, y) \geq \sigma_{g,l}(x, y)$ and $l\sigma_{g,l}(x, y) \geq \sigma_{f,k}(x, y)$ for all $x \in X$. This is $\sigma_{f,k} \equiv \sigma_{g,l}$. ■

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