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On invariance functions in Relativity Theory

J. Chargoy-Corona[1]
J. H. Arredondo R.[2]
[1,2]Departament of Mathematics
Division of Basic Sciences and Engineering
Metropolitan Autonomous University
Av. San Rafael Atlixco 186, Col Vicentina. Del Iztapalapa.
Zip Code 09340. Mexico City, Mexico

Abstract.

A new result for equivariant functions in terms of invariant functions in the case of Minkowski space is given. This generalizes the work of Hall and Wightman in the sense that only equivariance is required. In particular, it implies the possibility of defining physical magnitudes independently of the choice of the coordinate system, like the center of mass for relativistic particles.

Keywords: Lorentz group; equivariant function; invariant function.

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1 INTRODUCTION

There are several themes of investigation where the theory of equivariant and invariant functions are essential. Since these type of functions appear in quite different settings, they have particular properties according to the context under study. It is therefore mathematical interesting and of practical interest to characterize them.

In the subject of representation of the rotation group, invariance implies independence of equations and of measurable quantities with respect to the coordinate system, in the context of non-relativistic mechanics. For the case of the Lorentz group, invariance gives the same conclusion in relativistic mechanics [1] [3].

In the paper by D. Hall and A. S. Wightman [2] it has been proven that any complex valued function of n four-vector complex variables $z_j = \eta_j - i\xi_j,\ j = 1,\ldots,n$ which is *invariant* under the orthochronous Lorentz group, then it is an analytic function of the scalar products $z_j \cdot z_k,\ j,k = 1,\ldots,n$.

In this work we consider a four vector valued function that is *equivariant* with respect to the Lorentz group. We obtain explicit representations of the functions without assuming any kind of continuity.

In next article we look at some applications to quantum mechanics.

2 PRELIMINARIES

We denote $L(\mathbb{R}^n)$ as the linear transformations on $\mathbb{R}^n$, which we identify with the real square matrices of size *n*. I_n denotes the identity in $L(\mathbb{R}^n)$. The generalized orthogonal group $O(1,n)$ is by definition the subset of $L(\mathbb{R}^{n+1})$ given by

$$O(1,n) := \{g : J_{1,n} = g^T J_{1,n} g\}, \tag{1}$$

and the $(n+1)$-square matrix

$$((J_{1,n})_{\mu\nu}) = J_{1,n} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ \vdots & & -I_n & \\ 0 & & & \end{bmatrix}. \tag{2}$$

One can see that $g \in O(1,n)$ implies $g^{-1}, g^T \in O(1,n)$. Suppose that

$$L(\mathbb{R}^{n+1}) \ni g = \begin{bmatrix} a & w^T \\ v & A \end{bmatrix},$$

with $a \in \mathbb{R}, v, w \in \mathbb{R}^n, A \in L(\mathbb{R}^n)$. Then,

$$\begin{aligned} g \in O(1,n) &\Leftrightarrow (|v|^2 = a^2 - 1, A^T v = aw \text{ and } A^T A - ww^T = I_n) \\ &\Leftrightarrow (|w|^2 = a^2 - 1, Aw = av \text{ and } AA^T - vv^T = I_n). \end{aligned} \tag{3}$$

Here $|\cdot|$ denoting the norm in $\mathbb{R}^n$. $O(1,n)$ is a group under the multiplication in $L(\mathbb{R}^{n+1})$, and acts on $\mathbb{R}^{n+1}$ by left multiplication. One defines the orbit of the point $x \in \mathbb{R}^{n+1}$ as

$$O(1,n)\, x := \{g \cdot x : g \in O(1,n)\}.$$

Also, the isotropy or stability subgroup of x is defined to be

$$Iso(x, O(1,n)) := \{g \in O(1,n) : g \cdot x = x\} \tag{4}$$

For fixed $n > 2$, we denote $\vec{e}_j$, for $j = 1, \ldots, n$, as the usual canonical vectors in $\mathbb{R}^n$. Moreover,

$$\begin{aligned} e_o &:= (1, \vec{0}) \in \mathbb{R}^{1+n}, \\ e_j &:= (0, \vec{e}_j) \in \mathbb{R}^{1+n}, j = 1, \ldots, n. \end{aligned} \tag{5}$$

We have the following set identities.

$$\begin{aligned} O(1,n)\, e_o &= h_+ := \{(a,v) \in \mathbb{R} \times \mathbb{R}^n : \quad a^2 - |v|^2 = 1\} \\ O(1,n) e_1 &= h_- := \{(a,v) \in \mathbb{R} \times \mathbb{R}^n : \quad a^2 - |v|^2 = -1\} \\ O(1,n)(e_o + e_1) &= \mathrm{C} := \{(a,v) \in \mathbb{R} \times \mathbb{R}^n : \quad a = |v| > 0\} \\ O(1,n)(0) &= \{0\} \subset \mathbb{R}^{1+n}. \end{aligned} \tag{6}$$

Also, one obtains directly from definition that $g \in Iso(e_o, O(1,n))$ iff

$$g = \begin{bmatrix} 1 & 0^T \\ 0 & A \end{bmatrix}, \text{ for some orthogonal matrix } A.$$

We write this as

$$O(n) \cong Iso(e_o, O(1,n)). \tag{7}$$

Similarly, one obtains

$$O(1,n-1) \cong Iso(e_1, O(1,n)). \tag{8}$$

by means of the isomorphism

$$O(1,n-1) \ni \begin{bmatrix} a & w^T \\ v & A \end{bmatrix} \leftrightarrow \begin{bmatrix} a & 0 & & w^T & \\ 0 & 1 & 0 & \dots & 0 \\ & 0 & & & \\ v & \vdots & & A & \\ & 0 & & & \end{bmatrix} \tag{9}$$

Minkowski space

The Minkowski space-time is defined to be as the vector space R^4 with the bilinear form $B(\cdot,\cdot) : \mathrm{R}^4 \times \mathrm{R}^4 \to \mathrm{R}$, given by the formula:

$$B(x,y) = x^T \cdot J_{1,3} \cdot y. \tag{10}$$

It follows that $O(1,3)$ is the group of elements $g \in L(R^4)$ leaving invariant B:

$$B(gx, gy) = B(x,y) \quad (\forall\, x, y \in \mathrm{R}^4) \Leftrightarrow g \in O(1,3), \tag{11}$$

and called the Lorentz group which we denote as

$$£ := O(1,3).$$

The group £ acts on $\mathrm{R}^4 \times \mathrm{R}^4$ by

$$\mathrm{R}^4 \times \mathrm{R}^4 \ni (x,y) \mapsto (gx, gy) \quad (g \in £).$$

For given $\xi \in R^4 \times R^4$, the orbit of ξ under the action of £ is defined to be

$$£\xi := \{ g \cdot \xi : g \in £ \} \subset \mathrm{R}^4 \times \mathrm{R}^4.$$

We take $e_o = (1, \vec{0}),\ e_j = (0, \vec{e}_j),\ j = 1,2,3$ as the usual canonical vectors in R^4, and

$$H_{\pm} := \{ x \in R^4 : \ \pm B(x,x) > 0 \}, \quad H := H_- \cup H_+ \subset R^4,$$

also with

$$M_{1,1} = \{ (\mu e_o, \alpha e_o + \beta e_1) : \beta, \mu > 0, \alpha \in \mathrm{R} \} \subset H_+ \times \mathrm{R}^4$$
$$M_{1,2} = \{ (\mu e_o, \alpha e_o) : \mu > 0, \alpha \in \mathrm{R} \} \subset H_+ \times \mathrm{R}^4$$
$$M_{2,1} = \{ (\mu e_1, \beta e_o + \alpha e_1) : \beta, \mu > 0,\ \alpha \in \mathrm{R} \} \subset H_- \times \mathrm{R}^4$$

(12)

$$M_{2,2}=\{(\mu e_1,\alpha e_1+\beta e_2):\ \beta,\mu>0,\alpha\in \mathrm{R}\}\subset H_-\times \mathrm{R}^4$$
$$M_{2,3}=\{(\mu e_1,e_o+\alpha e_1+e_2):\alpha\in R,\ \mu>0\}\subset H_-\times R^4$$
$$M_{2,4}=\{(\mu e_1,\alpha e_1):\ \mu>0,\alpha\in \mathrm{R}\}\subset H_-\times \mathrm{R}^4$$

It is easy to see that

$$M_{i,j}\cap M_{k,l}=\begin{cases} M_{i,j}, & i=k;\ j=l\\ \varnothing, & otherwise\end{cases} \tag{13}$$

We put $M_1=M_{1,1}\cup M_{1,2}$ and $M_2=\bigcup_{j=1}^{4}M_{2,j}$.

Definition 1

(i) A function $f:H\times \mathrm{R}^4\to Y$ is said to be Lorentz-invariant (£ -invariant) iff $f(g\cdot x,g\cdot y)=f(x,y)\quad (\forall(x,y)\in H\times R^4,\forall g\in £)$.

(ii) A vector valued function $F:H\times \mathrm{R}^4\to \mathrm{R}^4$ is called equivariant iff

$$F(g\cdot x,g\cdot y)=g\cdot F(x,y)\quad (\forall(x,y)\in H\times R^4,\forall g\in £).$$

Lemma 1 *The following relations hold true.*

(i) $£M_1=H_+\times \mathrm{R}^4$

(ii) $£M_2=H_-\times \mathrm{R}^4$

(iii) $M_1\cap £M_2=M_2\cap £M_1=\varnothing$

Proof: For given $(x,y)\in H\times \mathrm{R}^4$ we define

$$\begin{aligned}\mu&=\mu(x):=|B(x,x)|^{\frac{1}{2}}>0\\ \alpha&=\alpha(x,y):=\frac{B(x,y)}{\mu}\\ \beta&=\beta(x,y):=|B(y,y)-\frac{B(x,y)^2}{B(x,x)}|^{\frac{1}{2}}\end{aligned} \tag{14}$$

Note that

$$v:=y-\frac{B(x,y)}{B(x,x)}x\quad\Rightarrow\quad B(v,x)=0. \tag{15}$$

Also,

$$B(v,v)=B(y,y)-\frac{B(x,y)^2}{B(x,x)}=sgn(B(v,v))\beta^2. \tag{16}$$

Because of (6), given $x\in H$, there exist $g_o,g_1\in £$ such that

$$x = \begin{cases} g_o \cdot (\mu e_o), & x \in H_+ \\ g_1 \cdot (\mu e_1), & x \in H_- \end{cases} \tag{17}$$

By using (14) and (15), we write for $p = 0,1$,

$$y = \nu + \alpha g_p \cdot e_p \quad or \quad g_p^{-1} \cdot y = g_p^{-1} \cdot \nu + \alpha e_p. \tag{18}$$

If $(t_o, t_1, t_2, t_3) = g_p^{-1} \cdot \nu$, from (15) we get

$$(-1)^p t_p = B(g_p^{-1}\nu, e_p) = \frac{B(\nu, x)}{\mu} = 0. \tag{19}$$

Suppose now $p = 0$: This means $x \in H_+$ and $g_o^{-1} \cdot \nu = (0, t_1, t_2, t_3)$. Equation (16) yields $\beta = \sqrt{t_1^2 + t_2^2 + t_3^2} \geq 0$. Owing to (7), we can take $k \in Iso(e_o, £)$ such that $k \cdot (\beta e_1) = (0, t_1, t_2, t_3)$. Then, (18) gives

$$g_o^{-1} \cdot y = (\alpha, t_1, t_2, t_3) = k \cdot (\alpha e_o + \beta e_1).$$

Therefore,

$$(x, y) = (g_o \cdot (\mu e_o), g_o \circ k \cdot (\alpha e_o + \beta e_1)) = g_o \circ k \cdot (\mu e_o, \alpha e_o + \beta e_1) \in £M_1.$$

This implies assertion (i) of the lemma.

When $p = 1$, one has $x \in H_-$ and $g_1^{-1}\nu = (t_o, 0, t_2, t_3)$, so that (18) becomes

$$g_1^{-1} \cdot y = g_1^{-1} \cdot \nu + \alpha e_1 = (t_o, \alpha, t_2, t_3). \tag{20}$$

By using (8) with $n = 3$, we choose $k_1, k_2, k_3 \in Iso(e_1, £)$, obeying

$$(t_o, 0, t_2, t_3) = \begin{cases} k_1 \cdot (\beta e_o), & t_o^2 - t_2^2 - t_3^2 > 0; \\ k_2 \cdot (\beta e_2), & t_o^2 - t_2^2 - t_3^2 < 0; \\ k_3 \cdot (e_o + e_2), & t_o^2 = t_2^2 + t_3^2 > 0. \end{cases} \tag{21}$$

This, together with (20), implying

$$g_1 \circ k_j \cdot (\mu e_1) = x \quad for \quad j = 1,2,3;$$

$$y = \begin{cases} g_1 \circ k_1 \cdot (\beta e_o + \alpha e_1), & t_o^2 - t_2^2 - t_3^2 > 0; \\ g_1 \circ k_2 \cdot (\alpha e_1 + \beta e_2), & t_o^2 - t_2^2 - t_3^2 < 0; \\ g_1 \circ k_3 \cdot (e_o + \alpha e_1 + e_2), & t_o^2 = t_2^2 + t_3^2 > 0; \\ (0,0,0,0), & t_o^2 = t_2^2 + t_3^2 = 0. \end{cases}$$

Denoting $k_4 := I_4$, we conclude from these identities that

$$(g_1 \circ k_j)^{-1} \cdot (x, y) = (\mu e_1, k_j^{-1} \circ g_1^{-1} \cdot y) \in M_{2,j},$$

for $j = 1,2,3,4.$ This proves (ii), and (iii) follows from $(i) - (ii)$. Δ

Remarks

(a). Functions μ,α,β defined in equation (14) are indeed £-invariant, with domain $H \times \mathrm{R}^4$.
(b). Any function $f : M_{i,j} \to \mathrm{R}$, can be expressed in the form $f = \tilde{f}(\mu,\alpha,\beta)$ where $\tilde{f}$ has domain contained in $\mathrm{R}^+ \times \mathrm{R} \times [0,\infty)$.

In order to analyze the homogeneous space structure of the orbits $£\xi$ for $\xi \in M_1 \cup M_2$, and to give appropriate parametrizations of them we need the following proposition.

Lemma 2 *Let* $M_{i,j}$, $i = 1,2,\ j = 1,\ldots,4,$ *be given by (12). Suppose* $g \cdot \xi \in M_1 \cup M_2$ *for some* $\xi \in M_{i,j}$ *and some* $g \in £$. *Then* $g \cdot \xi = \xi$ *and* $g \in E_{i,j}$, *where*

$$E_{i,j} = \begin{cases} Iso(e_o,£) \cap Iso(e_1,£) & i = 1,\ j = 1; \\ Iso(e_o,£), & i = 1, j = 2; \\ Iso(e_o,£) \cap Iso(e_1,£), & i = 2, j = 1; \\ Iso(e_1,£) \cap Iso(e_2,£), & i = 2, j = 2; \\ Iso(e_1,£) \cap Iso(e_o + e_2,£), & i = 2, j = 3; \\ Iso(e_1,£), & i = 2, j = 4. \end{cases}$$

Proof: First suppose that $g \cdot e_j = c \cdot e_j$ for some constant $c \in R$ and some $j = 0,1,2,3.$ Applying B on this equation, using (11), gives $c = \pm 1.$ Now we consider two cases.

(a). $(i = 1)$ and $(j = 1,2)$: Suppose $\xi \in M_1$. It follows from lemma 1 that $g \cdot \xi \in M_1$, and $\xi = (\mu e_o, \alpha e_o + \beta e_1)$, implying

$$g \cdot \xi = (\mu' e_o, \alpha' e_o + \beta' e_1)$$

for some $\mu, \mu' > 0;\ \alpha, \alpha' \in \mathrm{R};\ \beta, \beta' \ge 0.$ Therefore,

$$g e_o = \frac{\mu'}{\mu} e_o \Rightarrow \mu' = \mu \text{ and } g \in Iso(e_o,£). \tag{22}$$

Furthermore,

$$\alpha' e_o + \beta' e_1 = g \cdot (\alpha e_o + \beta e_1) = \alpha e_o + \beta\, g \cdot e_1. \tag{23}$$

Yielding,

$$\alpha' = B(\alpha' e_o + \beta' e_1, e_o) = B(\alpha e_o + \beta e_1, g^{-1} e_o) = B(\alpha e_o + \beta e_1, e_o) = \alpha.$$

Now we get from (23) that $\beta\, g \cdot e_1 = \beta' e_1 \Rightarrow \beta = \beta'$. In the case $\beta = 0$ we get

$$\xi \in M_{1,2} \text{ and } g \in Iso(e_o, \pounds).$$

So that $g \in E_{1,2}$. For the case $\beta > 0$ we obtain,

$$\xi \in M_{1,1} \text{ and } g \in Iso(e_o, \pounds) \cap Iso(e_1, \pounds),$$

yielding $g \in E_{1,1}$ and proving the statement of the proposition for $i = 1$ and $j = 1,2$.

(b). $(i = 2)$ and $(j = 1)$: Take $\xi \in M_2$. Lemma 1 shows that $g \cdot \xi \in M_2$.

Suppose now $\xi \in M_{2,1}$: this gives $\xi = (\mu e_1, \beta e_o + \alpha e_1)$, and $g \cdot \xi = (\mu' e_1, t_o e_o + t_1 e_1 + t_2 e_2)$, with $\mu, \beta, \mu' > 0$. Due to $g e_1 = \dfrac{\mu'}{\mu} e_1$, this leads to $\mu = \mu'$ and $g e_1 = e_1$. So that, $g \in Iso(e_1, \pounds)$. Furthermore, we have that

$$t_o e_o + t_1 e_1 + t_2 e_2 = g \cdot (\beta\, e_o + \alpha\, e_1) = \beta\, g \cdot e_o + \alpha\, e_1. \tag{24}$$

Then,

$$-t_1 = B(t_o e_o + t_1 e_1 + t_2 e_2, e_1) = B(g(\beta e_o + \alpha e_1), e_1) = B(\beta e_o + \alpha e_1, e_1) = -\alpha.$$

This equality and (24) give

$$\beta\, g \cdot e_o = t_o e_o + t_2 e_2 \text{ and } \beta^2 = t_o^2 - t_2^2. \tag{25}$$

In case $g \cdot \xi \in M_{2,j}$ for $j = 2,3$ or $j = 4$, we obtain, respectively, that

$$t_o = 0 \text{ and } t_2 > 0, t_o = t_2 = 1 \text{ or } t_o = t_2 = 0.$$

This leads in either case to $\beta^2 \leq 0$, obtaining necessarily that,

$$M_2 \ni g \cdot \xi = t_o e_o + t_1 e_1 \in M_{2,1} \Rightarrow t_o > 0, t_2 = 0.$$

Therefore, (25) implies that $t_o = \beta$ and $g e_o = e_o$, yielding $g \in Iso(e_1, \pounds) \cap Iso(e_o, \pounds)$. This proves the statement of the proposition in the case $i = 2$ and $j = 1$.

Remaining cases, namely, $i = 2$ and $j = 2,3,4$ follow by similar arguments. Δ

The previous lemmas and remarks yield the next corollary.

Corollary 1

(i) The set $M := M_1 \cup M_2$ contains just one point of each orbit.

(ii) The canonical projection

$$P : H \times \mathrm{R}^4 \to H \times \mathrm{R}^4/£$$

given by

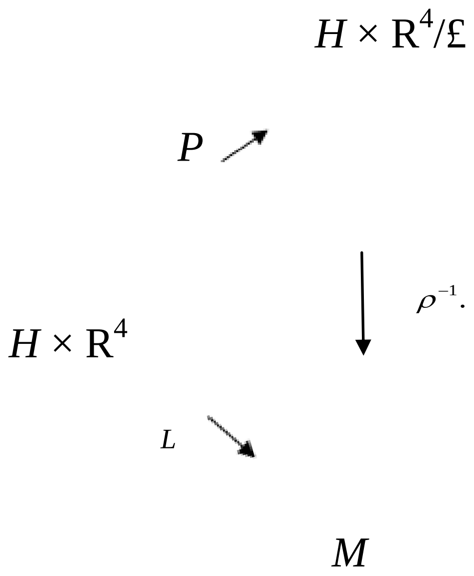
$$P(x, y) = £(x, y),$$

gives rise to a bijection ρ when restricted to M:

$$\rho : M \to H \times \mathrm{R}^4/£$$

$$\rho := P|_M .$$

(iii) The following diagram commutes:



and L is an $£$-invariant map. Furthermore, a function $h : H \times R^4 \to R$ is $£$- invariant iff there exists a function $f : M \to R$ such that

$$h(x, y) = f \circ L(x, y).$$

Proof: Statements (i) and (ii) follow from Lemma 1 and Lemma 2.

To prove (iii), note that invariance of L follows easily. Now suppose that h is $£$-invariant. Since $L|_M : M \to M$ is the identity map, one gets that statement (iii) holds true on M:

$$h(\xi) = h \circ L(\xi) \quad (\forall \xi \in M).$$

So that, we can take $f := h|_M$. For arbitrary $(x, y) \in H \times \mathrm{R}^4$, one can write $(x, y) = g \cdot (\xi_1, \xi_2)$, for some $g \in £$ and $(\xi_1, \xi_2) \in M$. Invariance of h and L yields,

$$h(x, y) = f \circ L(\xi_1, \xi_2) = f \circ L(g \cdot (\xi_1, \xi_2)) = f \circ L(x, y). \qquad \Delta$$

Theorem 1. *A function* $F : H \times \mathrm{R}^4 \to \mathrm{R}^4$ *is equivariant iff there exist* £*-invariant functions* $\lambda_p,\ p = 1,2,$ *such that for arbitrary* $(x, y) \in H \times \mathrm{R}^4$,

$$F(x, y) = \lambda_1(x, y)x + \lambda_2(x, y)y. \tag{26}$$

Proof: When F obeys equation (26), then F is equivariant.

Suppose now that F is equivariant. The proof is given in the following two steps:

1. We analyze the function F on M, and show that the statements of theorem hold true on this subset. Let us define the following one-parameter subgroups of £,

$$\phi_{i,j} : (R,+) \to E_{i,j},$$

$$\phi_{11}(t) = \phi_{12}(t) = \phi_{21}(t) = e^{tA_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix},$$

$$\phi_{22}(t) = e^{tA_2} = \begin{bmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{bmatrix},$$

$$\phi_{24}(t) = \phi_{23}(t) = e^{tA_3} = \begin{bmatrix} 1+\frac{t^2}{2} & 0 & -\frac{t^2}{2} & t \\ 0 & 1 & 0 & 0 \\ \frac{t^2}{2} & 0 & 1-\frac{t^2}{2} & t \\ t & 0 & -t & 1 \end{bmatrix}$$

with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = A_4 = A_2 - A_1.$$

Then, the infinitesimal generator of $\phi_{i,j}$ lies in the Lie algebra of £ [4]. Furthermore, equivariance implies, for $t \in \mathrm{R}$, and Q any of $A_p,\ p = 1,2,3,4,$

$$F(e^{tQ}x, e^{tQ}y) = e^{tQ}F(x, y).$$

We will denote $\xi = (\xi_1, \xi_2)$ in this proof. When $\xi \in M_{i,j}$ and ϕ_{ij} the corresponding group homomorphism in $E_{i,j}$, from lemma 2 we obtain the following identity for $t \in \mathrm{R}$,

$$F(\xi) = e^{tQ}F(\xi) \Rightarrow F(\xi) \in \ker Q.$$

For $i=1, j=1,2$, we choose $\phi_{ij}(t)=e^{tQ}=e^{tA_1}$, and ker Q is the subspace generated by $\{e_o, e_1\}$. Therefore, $\xi \in M_1$ implies

$$ker\, A_1 \ni F(\xi) = f_o(\xi)e_o + f_1(\xi)e_1 \tag{27}$$

for some functions f_o, f_1 and

$$\xi = (\mu e_o, \alpha e_o + \beta e_1); \quad \mu > 0, \beta \geq 0, \alpha \in \mathrm{R}.$$

For $\beta > 0$ we can write

$$\begin{aligned} F(\xi) &= \frac{1}{\mu}(f_o(\xi) - \frac{\alpha f_1(\xi)}{\beta})\xi_1 + \frac{f_1(\xi)}{\beta}\xi_2 \\ &\equiv f_{11}^{(1)}(\xi)\xi_1 + f_{11}^{(2)}(\xi)\xi_2, \quad (\forall \xi \in M_{1,1}). \end{aligned} \tag{28}$$

When $\beta = 0$, we denote a permutation $k \in \mathrm{O}(3) \cong Iso(e_o, \text{£}) = E_{1,2},$, interchanging just $e_1 \leftrightarrow e_2$. From (27) and lemma 2 one can deduce that

$$f_o(\xi)e_o + f_1(\xi)e_1 = F(\xi) = F(k\cdot\xi) = k\cdot(f_o(\xi)e_o + f_1(\xi)e_1) = f_o(\xi)e_o + f_1(\xi)e_2.$$

So that, $f_1(\xi) = 0$ for $\xi \in M_{1,2}$. This implies,

$$\begin{aligned} F(\xi) &= \frac{f_o(\xi)}{\mu}\xi_1 \\ &\equiv f_{12}^{(1)}(\xi)\xi_1 + f_{12}^{(2)}(\xi)\xi_2, \quad (\forall \xi \in M_{1,2}). \end{aligned} \tag{29}$$

Therefore, we have maps

$$f_{11}^{(1)}, f_{11}^{(2)} : M_{1,1} \to R \quad \text{and} \quad f_{12}^{(1)}, f_{12}^{(2)} : M_{1,2} \to R$$

obeying (28) and (29), respectively. A similar proceeding shows the existence of pair of functions,

$$f_{2j}^{(1)}, f_{2j}^{(2)} : M_{2,j} \to R,$$

obeying on $M_{2,j}$ for $1 \leq j \leq 4$,

$$F(\xi_1, \xi_2) = f_{2j}^{(1)}(\xi_1, \xi_2)\xi_1 + f_{2j}^{(2)}(\xi_1, \xi_2)\xi_2. \tag{30}$$

We consider maps $\tilde{\lambda}_p, p = 1,2,$ with domain M, defined by

$$\tilde{\lambda}_p(\xi) = \begin{cases} f_{1j}^{(p)}(\xi), & \xi \in M_{1,j};\ j = 1,2; \\ f_{2j}^{(p)}(\xi), & \xi \in M_{2,j};\ j = 1,2,3,4. \end{cases} \tag{31}$$

This yields, due to (28), (29) and (30),

$$F(\xi) = \tilde{\lambda}_1(\xi)\xi_1 + \tilde{\lambda}_2(\xi)\xi_2 \quad (\forall \xi \in M). \tag{32}$$

2. Equivariance of F, gives for arbitrary $(x, y) \in H \times \mathrm{R}^4$, with $(x, y) = g \cdot \xi = g \cdot (\xi_1, \xi_2)$ for some $g \in £$, $\xi \in M$,

$$g^{-1} \circ F(x, y) = F(\xi) = \tilde{\lambda}_1(\xi)\xi_1 + \tilde{\lambda}_2(\xi)\xi_2.$$

Yielding,

$$F(x, y) = \tilde{\lambda}_1(\xi)x + \tilde{\lambda}_2(\xi)y \quad (\forall (x, y) \in H \times \mathrm{R}^4). \tag{33}$$

Now we note that the maps (31) can be extended to the whole space $H \times \mathrm{R}^4$ as £-invariant functions λ_p, $p = 1,2$, by means of the formula:

$$\lambda_p : H \times R^4 \to R$$
$$\lambda_p(x, y) := \tilde{\lambda}_p \circ L(x, y)$$

This and equation (33) prove the proposition. Δ

3 System of two relativistic particles

We consider a system of two relativistic particles with positions x and y, and respective momentum

$$p = (p_o, \vec{p}) = (p_o, p_1, p_2, p_3) \qquad q = (q_o, \vec{q}) = (q_o, q_1, q_2, q_3).$$

Their total momentum is defined to be

$$P = p + q.$$

If one wants to define the relative energy-momentum vector $\varepsilon \equiv \varepsilon(p, q)$ of the system, it might be defined so that it is an equivariant function. From previous theorem, it must have the expression

$$\varepsilon = \lambda_1 p + \lambda_2 q \tag{34}$$

for some invariant functions $\lambda_j \equiv \lambda_j(p, q)$, $j = 1,2$. In similarity with classical mechanics, one can impose the condition

$$\lambda_1 - \lambda_2 = 1, \tag{35}$$

so that

$$\varepsilon = \lambda_1 P - q, \qquad \lambda_1(p, q) = \frac{B(P, \varepsilon) + B(P, q)}{B(P, P)}. \tag{36}$$

In particular, when $B(P,\varepsilon)=0$, one gets the energy-momentum vector of the system as

$$\lambda_1=\frac{B(P,q)}{B(P,P)} \quad \Leftrightarrow B(P,\varepsilon)=0 \Leftrightarrow \varepsilon=\frac{B(P,q)}{B(P,P)}p-\frac{B(P,p)}{B(P,P)}q. \tag{37}$$

By using (2), the vector ε in this case is coupled to the vector $\Upsilon = x-y$ in the sense that the Poisson bracket obeys

$$\{\Upsilon_\mu,\varepsilon_\nu\}=(J_{1,3})_{\mu\nu}.$$

Furthermore, for

$$X:=\frac{B(P,q)}{B(P,P)}x+\frac{B(P,p)}{B(P,P)}y, \tag{38}$$

one gets

$$\{X_\mu,P_\nu\}=(J_{1,3})_{\mu\nu}.$$

P being the total momentum, then X can be thought of as the center of mass-energy of the system. It follows that once the relative energy-momentum ε is fixed, also the center of mass X is determined except on a set of measure zero in $\mathrm{R}^4\times\mathrm{R}^4$.

See [5] for a further discussion about this example.

We show an additional application of our results in the following example.

Example 1

Let us denote the Haar measure on the Lorentz group £ by μ. For a given function $f: H\times R^4 \to R$, we consider the following function defined by the (Lebesgue) integral on £:

$$F_a: H\times R^4\to R^4$$

$$F_a(x,y):=\int_{£} f(g^{-1}\cdot(x,y))g\cdot a d\mu(g). \tag{39}$$

Here $a\in R^4$ is a fixed vector. It follows from invariance of Haar measure that F_a is equivariant [6] [7]. Assuming convergence of the integral, our result implies that there exist £-invariant functions $\lambda_{i,a}, i=1,2$, such that

$$F_a(x,y)=\lambda_{1,a}(x,y)x+\lambda_{2,a}(x,y)y.$$

Theorem 1 gives a method to fix F_a. In fact, the result implies that it is sufficient to know $\lambda_{1,a}$ and $\lambda_{2,a}$ on M. This in turn implies only a few calculations. For instance, for $\xi\in M_{1,1}\subset M$, then $\xi=(\mu e_o,\alpha e_o+\beta e_1)$ and

$$\begin{aligned} B(F_a(\xi),e_1) = B(\lambda_{1,a}\xi_1+\lambda_{2,a}\xi_2,e_1) &= \lambda_{2,a}B(\xi_2,e_1)\\ &= -\beta\lambda_{2,a} \\ &\Rightarrow \lambda_{2,a}=\lambda_{2,a}(\xi)=-\frac{B(F_a(\xi),e_1)}{\beta}, \end{aligned} \tag{40}$$

and giving for $\lambda_{1,a}(\xi),\xi \in M_{1,1}$, the formula

$$\lambda_{1,a}(\xi) = \frac{1}{\mu} B(F_a(\xi), e_o + \frac{\alpha}{\beta} e_1).$$

In the same way, one can determine $\lambda_{1,a},\lambda_{2,a}$ on the whole set M. This fixes F_a on its whole domain.

We mention that there are several applications for the theory of equivariant functions, and where our results might be helpful. See for instance [8] [9] [10].

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References

[1] I. M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro. *Representations of the Rotation and Lorentz groups and their Applications.* Pergamon, London; Macmillan, New York, 1963.

[2] D. Hall and A. S. Wightman. *A Theorem on Invariant Analytic Functions with Applications to Relativistic Quantum Field Theory*, E. Munksgaard, publisher. 1957.

[3] M. A. Naimark. *Linear Representations of the Lorentz Group*, (International Series of Monographs in Pure and Applied Mathematics, Vol. 63) Oxford/London/Edinburgh/New York/Paris/Frankfurt. Pergamon Press, 1964.

[4] Y. Matsushima. *Differentiable Manifolds*, (Series in Pure and Applied Mathematics, Vol 9) New York & Basel. Macel Dekker, Inc., 1972.

[5] A. Plis′, T. Zastawniak. *Relativistic Quantum Mechanics for two interacting particles in one-time representation*. Reports on Mathematical Physics, **31**, 317(1992).

[6] E. Hewitt, K. A. Ross. *Abstract Harmonic Analysis I*. Grundlehren der mathematischen Wissenchaften 115. Springer Verlag New York, Inc. 2nd Edition, 1979.

[7] A. Wawrzyñczyk. *Group Representations and Special Functions*. D. Reidel Publishing Company. Mathematics and Its Applications (East European Series). Dordrecht-Boston-Lancaster, 1984.

[8] M. Reisert, H. Burkhardt. *Learning Equivariant Functions with Matrix Valued Kernels*. Journal of Machine Learning Research **8**, 305(2007).

[9] A. Sebbar, A. Sebbar. *Equivariant functions and integrals of elliptic functions*. Geometriae Dedicata, **160**, 373(2012).

[10] A. G. Wasserman. *Equivariant Differential Topology*. Topology, **8**, 127(1969).