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On the Integer Zeros of Krawtchouk Polynomials of Degree 8

Ahmad M Alenezi

The Higher Institute of Telecommunications and Navigation
PAAET, Kuwait

Abstract.

Krawtchouk polynomials plays very important role in many different areas of mathematics such as discrete mathematics, coding theory, association schemes and graph theory. The issue of existence of integer zeros of Krawtchouk polynomials is crucial for the existence of combinatorial structures in the Hamming schemes. In this Paper our goal is to investigate the integer zeros of the modified Krawtchouk polynomials of the 8^{th} order, $Q_k^n(y) = k!\,P_k^n\left(\frac{n-y}{2}\right)$, where $k = 8$.

Keywords: Krawtchouk Polynomials; Integer Zeros; Modular Setting.

1. Integer Zeros of Krawtchouk Polynomials of Degree 6 and 7.

1.1. Krawtchouk Polynomials of Degree 6.

A complete sets of the integral zeros of the binary Krawtchouk polynomials of degree 6 and 7 has been provided by Roelof J. Stroker in his Paper [2].

Krawtchouk polynomials for degree 6 is:

$$y^6 - 15y^4n + 40y^4 + 45y^2n^2 - 210y^2n - 15n^3 + 184y^2 + 90n^2 - 120n = 0 \qquad (1)$$

Using Diophantine Equation to solve (1).

Let $U = n$ and $V = y^2$ in (1), the following diophantine equation comes in:

$$-15U^3 + 45U^2V - 15UV^2 + V^3 + 90U^2 - 210UV + 40V^2 - 120U + 184V = 0 \qquad (2)$$

Solution of U and V in (2) was shown in details in [2].

1.2. Zeros of Krawtchouk Polynomials of Degree 6.

Theorem 1. [2] The diophantine equation (2) has integral solutions (U, V) as given in table 1, and no others.

Table 1: Solutions of equation (1)

Solution (U,V) of *(2)*, $U = n$ and $V = y^2, x \leq n/2$											
(U,V)	x	n	y	(U,V)	x	n	y	(U,V)	x	n	y
(-14,-56)				(3,1)	1	3	1	(12,4)	5	12	2
(-4, -20)				(3,9)	0	3	3	(12,36)	3	12	6
(-1,-9)				(4,0)	2	4	0	(12,100)	1	12	10
(0,0)	0	0	0	(4,4)	1	4	2	(16,144)	2	16	12
(1,1)	0	1	1	(4,16)	0	4	4	(25,9)	11	25	3
(2,-14)				(5,1)	2	5	1	(67,25)	31	67	5
(2,0)	1	2	1	(5,9)	1	5	3	(345,1225)	155	345	35
(2,4)	0	2	2	(5,25)	0	5	5				
(3,-5)				(9,25)	2	9	5				

In addition to the solutions, the table also gives the corresponding values of x, n, y. Symmetry about $x = n/2$ permits the restriction to $x \leq n/2$.

1.3. Krawtchouk Polynomials of Degree 7.

Modified Krawtchouk polynomials for degree 7 is:

$$y(y^6 - 21y^4n + 70y^4 + 105y^2n^2 - 630y^2n - 105n^3 + 784y^2 + 840n^2 - 1764n + 720) = 0 \quad (3)$$

Using Diophantine Equation to solve (3).

Let $U = n - 1$ and $V = y^2 - 1$ in (3), the following diophantine equation comes in:

$$-105U^3 + 105U^2V - 21UV^2 + V^3 + 630U^2 - 462UV + 52V^2 - 840U + 360V = 0 \quad (4)$$

Solution of U and V in (4) was shown in details in [2].

1.4. Zeros of Krawtchouk Polynomials of Degree 7.

Theorem 2. [2] The diophantine equation (4) has integral solutions (U,V) as given in table 2, and no others.

Table 2: Solutions of equation (3)

Solution (U,V) of *(3)*, $U = n$ and $V = y^2, x \leq n/2$											
(U,V)	x	n	y	(U,V)	x	n	y	(U,V)	x	n	y
(-22,-132)				(3,3)	1	4	2	(13,15)	5	14	4
(-6, -42)				(3,15)	0	4	4	(13,63)	3	14	8
(-3,-25)				(4,0)	2	5	1	(13,143)	1	14	12
(0,0)	0	1	1	(4,8)	1	5	3	(16,80)	4	17	9
(1,3)	0	2	2	(4,24)	0	5	5	(21,255)	4	22	16
(2,-18)				(5,3)	2	6	2	(1028,1368)	469	1029	37
(2,0)	1	3	1	(5,15)	1	6	4				
(2,8)	0	3	3	(5,35)	0	6	6				
(3,-7)				(8,8)	3	9	3				

In addition to the solutions, the table also gives the corresponding values of x, n, y. Symmetry about $x = n/2$ permits the restriction to $x \leq n/2$.

The solution process employs recent developments in the estimation of linear forms in elliptic logarithms. Extensive coverage of this method is given in [3], [4], [5]. Proof of theorem 1 is found in [2]. The proof of theorem 2 has entirely similar structure.

2. Integer Zeros of Krawtchouk Polynomials of Degree 8.

Modified Krawtchouk polynomials for degree *8*, $Q_k^n(y) = k!\,P_k^n\left(\frac{n-y}{2}\right)$, where $k = 8$.

$$Q_8^n(y) = y^8 - 28y^6n + 112y^6 + 210y^4n^2 - 1540y^4n - 420y^2n^3 + 2464y^4 + 4200y^2n^2 + 105n^4 - 11872y^2n - 1260n^3 + 8448y^2 + 4620n^2 - 5040n = 0 \quad (5)$$

2.1. Modular Setting on Krawtchouk Polynomials of Degree *8.*

Consider equation (5) with $n, y \in \mathbb{Z}$. Further $f(n,y)$ stands for the right hand of the equation.

(1) Considering (5) *(mod 7)*, build matrix $(f(n,y)\ mod\ 7)_{0\le n\le 6,\ 0\le y\le 6}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}$$

We can see that $f(n,y)\ mod\ 7 = 0$ if and only if $n = 0$ *(y is arbitrary)*, so integer zeros (n,y) of (5) must be of the form $n = 7m,\ m \in \mathbb{Z}$.

Remark 1. Condition $n = 7m,\ m \in \mathbb{Z}$ can be obtained without calculation $f(n,y)\ mod\ 7$ for 49 pairs (n,y). Remember the little theorem of Fermat:

$$a^p = a \ (mod\ p).$$

For any simple p and integer a. Assuming $p = 7,\ a = y$, we obtain that $y^8 = y^2$. So equation (5) *(mod 7)* becomes very simple:

$$(4n^2)\ mod\ 7 = 0$$

and $n = 7m,\ m \in \mathbb{Z}$.

(2) Because $f(n,y)\ mod\ 7 = 0$ only when $n = 0$, we can substitute $n = 7m$ with $m \in \mathbb{Z}$. Separate the part of $f(n,y)$, divisible by 49:

$$\begin{aligned} f(n,y) &= y^8 - 28y^6n + 112y^6 + 210y^4n^2 - 1540y^4n - 420y^2n^3 + 2464y^4 + 4200y^2n^2 \\ &+ 105n^4 - 11872y^2n - 1260n^3 + 8448y^2 + 4620n^2 - 5040n \\ &= (y^8 + 14y^6 + 14y^4 + 20y^2) + (-28\cdot 7y^6m + 98y^6 + 210\cdot 7^2y^4m^2 \\ &- 1540\cdot 7y^4m - 420\cdot 7^3y^2m^3 + 2450y^4 + 4200\cdot 7^2y^2m^2 + 105\cdot 7^4m^4 \\ &- 11872\cdot 7y^2m - 1260\cdot 7^3m^3 + 8428y^2 + 4620\cdot 7^2m^2 - 5040\cdot 7m) \end{aligned} \quad (6)$$

Denote

So,

$$g(m,y) = \left(\frac{1}{49}\right)(-28\cdot 7y^6m + 98y^6 + 210\cdot 7^2y^4m^2 - 1540\cdot 7y^4m - 420\cdot 7^3y^2m^3 + 2450y^4 + 4200\cdot 7^2y^2m^2 + 105\cdot 7^4m^4 - 11872\cdot 7y^2m - 1260\cdot 7^3m^3 + 8428y^2 + 4620\cdot 7^2m^2 - 5040\cdot 7m) = -4y^6m + 2y^6 + 210y^4m^2 - 210y^4m - 420\cdot 7y^2 + 50y^4 + 4200y^2 + m^2 + 105\cdot 7^2m^4 - 1696y^2m - 1260\cdot 7^2m^3 + 172y^2 + 4620m^2 - 720m \tag{7}$$

we can rewrite (5) in the form:

$$y^8 + 14y^6 + 14y^4 + 20y^2 + 49g(m,y) = 0 \tag{8}$$

where $g(m,y)$ is a polynomials with integer coefficients . Obviously,

$$49g(m,y) mod\ 49 = 0$$

for any integer n, y. So, (8) holds $(mod\ 49)$ if and only if $y\ mod\ 7 = 0$ (i.e. $y^2\ mod\ 49 = 0$), or

$$(y^6 + 14y^4 + 14y^2 + 20)\ mod\ 49 = 0 \tag{9}$$

Considering (9) with $(mod\ 49)$ condition, we have such possible value for $y\ mod\ 49$.

$$y\ mod\ 49\ \in \{1,3,5,44,46,48\}$$

Note that $44 = -5, 46 = -3, 48 = -1\ (mod\ 49)$, respectively to evenness of right hand of (9) and (5) by y. So, we have such alternatives for y:

$$y = 7s, \qquad y = \pm 1 + 49s, \qquad y = \pm 3 + 49s, \qquad y = \pm 5 + 49s,$$

where $s \in \mathbb{Z}$.

3. Considering y^2 mod 9, we can obtain four alternatives:

$$y^2 = 9k, \qquad y^2 = 9k + 1, \qquad y^2 = 9k + 4, \qquad y^2 = 9k + 7 \tag{10}$$

where $k \in \mathbb{Z}$.

Now substitute y^2 in (5) with alternatives (10).

$y^2 = 9k + 1$ (it occurs when $y\ mod\ 9 = 1$ or $y\ mod\ 9 = 8$; note that $8 = -1\ (mod\ 5)$). All coefficients of $f(n, \sqrt{9k+1})$ is divisible by 3. Equation $\frac{f(n,\sqrt{9k+1})}{3} = 0$ can be written in the form:

$$25344k + 66528k^2 + 27216k^3 + 2187k^4 - 1680n - 35616kn - 41580k^2n - 6804k^3n + 1420n^2 + 12600kn^2 + 5670k^2n^2 - 420n^3 - 1260kn^3 + 35n^4 = 0 \tag{11}$$

Equation (11) with condition $(mod\ 9)$ has zeros (n, y) with $n = 3,$ and $n = 8$. So, we have such alternatives for n:

$$n = 9j + 3, \;\; n = 9j + 8,$$

where $j \in \mathbb{Z}$. Taking into account that $n = 7m$ $(m \in \mathbb{Z})$, we have such alternatives for n:

$$n = 63j + 21, \;\; n = 63j + 35,$$

where $j \in \mathbb{Z}$.

$y^2 = 9k + 4$ (it occurs when $y \, mod \, 9 = 2$ or $y \, mod \, 9 = 7$; note that $7 = -2 \, (mod \, 5)$). All coefficients of $f(n, \sqrt{9k+4})$ is divisible by 3. Equation $\frac{f(n,\sqrt{9k+4})}{3} = 0$ can be written in the form:

$$\begin{aligned} 26880 + 101376k + 105408k^2 + 31104k^3 + 2187k^4 - 26320n - 76608kn \\ - 50652k^2n - 6804k^3n + 8140n^2 + 17640kn^2 + 5670k^2n^2 \\ - 980n^3 - 1260kn^3 + 35n^4 = 0 \end{aligned} \quad (12)$$

Equation (12) with condition $(mod \, 9)$ has zeros (n, y) with $n = 3,$ and $n = 8$. So, we have such alternatives for n:

$$n = 9j + 5, \;\; n = 9j + 6,$$

where $j \in \mathbb{Z}$. Taking into account that $n = 7m$ $(m \in \mathbb{Z})$, we have such alternatives for n:

$$n = 63j + 14, \;\; n = 63j + 42,$$

where $j \in \mathbb{Z}$.

$y^2 = 9k + 7$ (it occurs when $y \, mod \, 9 = 4$ or $y \, mod \, 9 = 5$; note that $5 = -4 \, (mod \, 5)$). All coefficients of $f(n, \sqrt{9k+7})$ is divisible by 3. Equation $\frac{f(n,\sqrt{9k+7})}{3} = 0$ can be written in the form:

$$\begin{aligned} 73563 + 182340k + 137970k^2 + 34020k^3 + 2187k^4 - 57736n - 112644kn \\ - 57456k^2n - 6804k^3n + 14650n^2 + 21420kn^2 + 5670k^2n^2 \\ - 1400n^3 - 1260kn^3 + 35n^4 = 0 \end{aligned} \quad (13)$$

Equation (13) with condition $(mod \, 9)$ has zeros (n, y) with $n = 3,$ and $n = 8$. So, we have such alternatives for n:

$$n = 9j + 6, \;\; n = 9j + 8,$$

where $j \in \mathbb{Z}$. Taking into account that $n = 7m$ $(m \in \mathbb{Z})$, we have such alternatives for n:

$$n = 63j + 42, \;\; n = 63j + 35,$$

where $j \in \mathbb{Z}$.

$y^2 = 9k$ (obviously, sufficient to consider $y^2 = 9k^2$; it occurs when $y \, mod \, 3 = 0$). All coefficients of $f(n, 3k)$ is divisible by 3. Separating the part divisible by 9, rewrite equation $\frac{f(n,3k)}{3} = 0$ in the form:

$$3n + 6kn + 7n^2 + 3n^3 + 8n^4 + 9h(n,k) = 0 \quad (14)$$

where $h(n,k)$ is a polynomial with integer coefficients. Obviously, (14) holds $(mod\ 9)$ if $n\ mod\ 9 = 0$, or

$$(3n + 6kn + 7n^2 + 3n^3 + 8n^4)\ mod\ 9 = 0 \quad (15)$$

For equation (15) we have 27 zeros $(mod\ 9)$

n	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6	7	7	7	8	8	8
k	1	4	7	1	4	7	1	4	7	2	5	8	2	5	8	2	5	8	0	3	6	0	3	6	0	3	6

4. Conclusion.

The following conditions were obtained for (n, y) being integer zero of equation (5).

4.1. $n = 7m,\ \ m \in \mathbb{Z}$.

4.2. $y = 7s,\ \ y = \pm 1 + 49s,\ \ y \pm 3 + 49s,\ \ y \pm 5 + 49s,$
where $s \in \mathbb{Z}$.

4.3. One of the following alternatives must hold:

n	y	n	y	n	y	n	y
$63j + 21$	$9i \pm 1$	$63j$	$3(9i + 4)$	$63j + 56$	$3(9i + 7)$	$63j + 77$	$3(9i + 2)$
$63j + 35$	$9i \pm 1$	$63j$	$3(9i + 7)$	$63j + 21$	$3(9i + 2)$	$63j + 77$	$3(9i + 5)$
$63j + 14$	$9i \pm 2$	$63j + 28$	$3(9i + 1)$	$63j + 21$	$3(9i + 5)$	$63j + 77$	$3(9i + 8)$
$63j + 42$	$9i \pm 2$	$63j + 28$	$3(9i + 4)$	$63j + 21$	$3(9i + 8)$	$63j + 42$	$3(9i + 0)$
$63j + 42$	$9i \pm 4$	$63j + 28$	$3(9i + 7)$	$63j + 49$	$3(9i + 2)$	$63j + 42$	$3(9i + 3)$
$63j + 35$	$9i \pm 4$	$63j + 56$	$3(9i + 1)$	$63j + 49$	$3(9i + 5)$	$63j + 42$	$3(9i + 6)$
$63j$	$3(9i + 1)$	$63j + 56$	$3(9i + 4)$	$63j + 49$	$3(9i + 8)$	$63j + 7$	$3(9i + 0)$
$63j + 7$	$3(9i + 6)$	$63j + 35$	$3(9i + 0)$	$63j + 35$	$3(9i + 6)$	$63j + 7$	$3(9i + 3)$

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