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High Dimensional Schwartz Caudrey-Dobb-Gibbon System: Painlevé Integrability and Exact Solutions

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Abstract

The usual (1+1)-dimensional Schwartz Caudrey-Dobb-Gibbon equation is extended to the general $(n+1)$-dimensional system. A singularity structure analysis for the extension system is carried out. It demonstrates that the extension system admits the Painlevé property. The exact solutions for the extension system are obtained with the Painlevé-Bäcklund transformation. In the meanwhile, some properties of the soliton solutions for the extension system are shown by some figures.

Keywords: High-Dimensional Integrable System; CDG Equation; Schwartz Form; Painleve Analysis; Exact Solutions.

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Introduction

Modern soliton theory is widely applied in almost all the physics fields, such as field theory, condensed matter physics, plasma physics, optics, particle, nuclear physics, etc [1, 2]. However, most of the present studies of the soliton theory and soliton applications are restricted in (1+1) and (2+1)-dimensions. The real physical space is (3+1)-dimensional, one hopes to find some (3+1)-dimensional integrable models. To find high dimensional integrable models is one of the important problems in mathematical physics. There are several ways to obtain high dimensional equations [3, 4, 5, 6, 7, 8, 9, 10, 11]. It is said that the known (1+1) and (2+1) dimensional integrable models possess Schwartz form which is conformal invariant. The conformal invariant forms may be best the candidates in finding higher integrable systems. The high-dimensional Painlevé integrable models have been obtained with the Schwartz Korteweg-de Vries, Boussinesq and Kadomtsev-Petviashvili equations [9, 10, 11]. Naturally, we hope that the more Schwartz equations can be extended to high dimensional Painlevé integrable systems.

In this letter, we extend the Schwartz Caudrey-Dobb-Gibbon (CDG) equation to high dimension case. The usual CDG equation is [12]

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0. \qquad (1)$$

Since CDG equation possesses typical properties of a soliton equation, a great deal of research works on the CDG equation have been carried out. The properties of this equation such as the Bäcklund transformation, Lax pair, Painlevé property, the nonlinear superposition formula, muti-soliton have been found [12, 13, 16, 15, 14].

The rest of the present paper is organized as follow. The arbitrary dimensional Painlevé integrable system are constructed in section 2. In section 3, the exact solutions for the $(n+1)$-dimensional system are obtained by the Painlevé-Bäcklund transformation. Some behaviour of the soliton solutions for the extension system are studied with some figures. The last section contains some conclusions.

(N+1)-Dimensional CDG Extension

For the CDG equation, the Schwartz form is [13]

$$\frac{\phi_t}{\phi_x}+\frac{\partial^2}{\partial x^2}\{\phi;x\}+4\{\phi;x\}^2=0, \tag{2}$$

where $\{\phi;x\}=(\frac{\phi_{xx}}{\phi_x})_x-\frac{1}{2}(\frac{\phi_{xx}}{\phi_x})^2$ is the Schwartz derivative of ϕ. The usual CDG equation and its Schwartz form are related to each other by

$$u=\frac{\partial^2}{\partial x^2}\ln\phi-\frac{1}{6}\frac{\phi_{xxx}}{\phi_x}.$$

To extend the Schwartz CDG equation (2) in high dimension, we may take many forms. Here, we take an ($n+1$)-dimensional CDG extension as

$$\sum_{i=1}^{n}(a_i\frac{\phi_t}{\phi_{x_i}}+b_i\frac{\partial^2}{\partial {x_i}^2}\{\phi;x_i\}+4c_i\{\phi;x_i\}^2)=0, \tag{3}$$

where a_i, b_i, c_i $(i=1,2,\cdots,n)$ are constants. (3) turns into the usual Schwartz form with $a_i=b_i=c_i=0$ $(i=2,3,\cdots,n)$. Meanwhile, (3) is invariant under the Möbious transformation

$$\phi\to\frac{a+b\phi}{c+d\phi},\quad ad\neq bc.$$

In order to use the Weiss-Tabor-Carnevale (WTC) approach, we make the following transformation

$$\phi=\exp F,\quad u_0=F_t,\quad u_i=F_{x_i},\quad i=1,2,\cdots,n, \tag{4}$$

Substituting expressions (4) into (3), we get the following system

$$\sum_{i=1}^{n}[a_i\frac{u_0}{u_i}+b_i(-u_{i,x_i}^2-u_iu_{i,x_ix_i}+\frac{u_{i,x_ix_ix_ix_i}}{u_i}-\frac{4u_{i,x_ix_i}^2+5u_{i,x_i}u_{i,x_ix_ix_i}}{u_i^2}+\frac{17u_{i,x_i}^2u_{i,x_ix_i}}{u_i^3}) \\ +c_i(u_i^4+6u_{i,x_i}^2-4u_iu_{i,x_i,x_i}+\frac{4u_{i,x_ix_i}^2}{u_i^2}-\frac{12u_{i,x_i}^2u_{i,x_ix_i}}{u_i^3}+\frac{9u_{i,x_i}^4}{u_i^4})]=0, \tag{5}$$

$$u_{i,t}-u_{0,x_i}=0,\quad i=1,2,\cdots,n, \tag{6}$$

where (5b) is the compatibility condition of transformations (4).

Now, we use the standard WTC approach to prove the Painlevé property of (2). We effect a local Laurent expansion in the neighborhood of a non-characteristic singular manifold $\phi_1=0$. It assumes

$$u_i=\sum_{j=0}^{\infty}u_{ij}\phi_1^{j+\alpha_i},\quad i=0,1,\cdots,n, \tag{7}$$

where u_{ij} are analysis functions of (t,x_i) and α is integer to be determined. From the corresponding leading order analysis, we obtain

$$\alpha_i=-1,\quad u_{00}^2=\phi_{1,t}^2,\quad u_{i0}^2=\phi_{1,x_i}^2. \tag{8}$$

Substituting (7) and (8) into (2), we have

$$(j+1)(j-1)^{n+1}(j-2)(j-3)\sum_{i=1}^{n}b_iu_{i0}^4\ =\ f(u_{ik},i=0,1,\cdots,n,k\le j-1) \tag{9}$$

$$(j-1)(u_{0j}u_{i0}-u_{ij}u_{00})=u_{i(j-1),t}-u_{0(j-1),x_i}, \tag{10}$$

where f is a complicated function of $(u_{ik}, i=0,1,\cdots,n, k\le j-1)$ and the derivatives of the singularity manifold ϕ_1. The resonance points are located at

$$j=-1,\underbrace{1,1,\cdots,1}_{n+1},2,3. \tag{11}$$

The resonance at $j=-1$ corresponds to the arbitrary singularity manifold ϕ_1. At $n+1$ resonance $j=1$ and two resonance $j=2,3$, there are $n+3$ compatibility conditions

$$\sum_{i=1}^{n}[3b_i(\phi_{1,x_i}^4-u_{i0}^2\phi_{1,x_i}^2)+c_i(u_{i0}^4+\phi_{1,x_i}^4-2u_{i0}^2\phi_{1,x_i}^2)]=0, \tag{12}$$

$$u_{i0,t}-u_{00,x_i}=0,\quad i=1,2,\cdots,n, \tag{13}$$

$$\sum_{i=1}^{n}[b_i(\frac{2u_{i,x_i}\phi_{1,x_i}^3}{u_i}+4u_iu_{i,x_i}\phi_{1,x_i}+u_i^2\phi_{1,x_i}-7\phi_{1,x_i}^2\phi_{1,x_ix_i})$$
$$+c_i(\frac{4\phi_{1,x_i}^3u_{i,x_i}}{u_i}-4u_iu_{i,x_i}\phi_{1,x_i}-4\phi_{1,x_i}^2\phi_{1,x_ix_i}+4\phi_{1,x_ix_i}u_i^2)]=0, \tag{14}$$

$$\sum_{i=1}^{n}[b_i(\frac{2u_{i,x_i}^2\phi_{1,x_i}^2}{u_i^2}-\frac{2u_{i,x_ix_i}\phi_{1,x_i}^2}{u_i}-\frac{3u_{i,x_i}\phi_{1,x_i}\phi_{1,x_ix_i}}{u_i}+3\phi_{1,x_i}\phi_{1,x_ix_ix_i}-2\phi_{1,x_ix_i}^2-u_iu_{i,x_ix_i}-u_{i,x_i}^2)$$
$$+c_i(\frac{4u_{i,x_ix_i}\phi_{1,x_i}^2}{u_i}-\frac{2u_{i,x_i}^2\phi_{1,x_i}^2}{u_i^2}-4u_iu_{i,x_ix_i}+4\phi_{1,x_ix_i}^2+6u_{i,x_i}^2)]=0. \tag{15}$$

Fortunately, it is straightforward to see that the conditions (2) are satisfied identically using the results of (8). Therefore, the $(n+1)$ -dimensional Schwartz CGD system is integrable in the sense that it possesses the Painlevé property.

Traveling Wave Solution for (N+1)-Dimensional CDG Extension

The investigation of the traveling wave solutions of nonlinear evolution equations plays an important role in the study of nonlinear wave phenomena. It is considered to be the most effective and direct algebraic method for solving nonlinear equations [17]. Here, we shall study the traveling wave solution of the $(n+1)$ -dimensional CDG extension system. With the Painlevé-Bäcklund transformation

$$u_0=\frac{\phi_{1,t}}{\phi_1},\quad u_i=\frac{\phi_{1,x_i}}{\phi_1},\quad i=1,2,\cdots,n, \tag{16}$$

(2) can be simplified the follow form

$$\sum_{i=1}^{n}[a_i\frac{\phi_{1,t}}{\phi_{1,x_i}}+b_i(-\phi_{1,x_ix_i}^2-\phi_{1,x_i}\phi_{1,x_ix_ix_i}+\frac{\phi_{1,x_ix_ix_ix_ix_i}}{\phi_{1,x_i}}-\frac{4\phi_{1,x_ix_ix_i}^2+5\phi_{1,x_ix_i}\phi_{1,x_ix_ix_ix_i}}{\phi_{1,x_i}^2}+\frac{17\phi_{1,x_ix_i}^2\phi_{1,x_ix_ix_i}}{\phi_{1,x_i}^3})$$
$$+c_i(\phi_{1,x_i}^4+6\phi_{1,x_ix_i}^2-4\phi_{1,x_i}\phi_{1,x_ix_ix_i}+\frac{4\phi_{1,x_ix_ix_i}^2}{\phi_{1,x_i}^2}-\frac{12\phi_{1,x_ix_i}^2\phi_{1,x_ix_ix_i}}{\phi_{1,x_i}^3}+\frac{9\phi_{1,x_ix_i}^4}{\phi_{1,x_i}^4})]=0. \tag{17}$$

We assume the traveling wave solution

$$\phi_1 = \phi_1(\xi), \quad \xi = k(\sum_{i=1}^{n} x_i - ct), \tag{18}$$

where k and c are arbitrary constants to be determined. Substituting (18) into (??), we can easily find that the equation is fully satisfied when

$$a_i = b_i = c_i, \quad \sum_{i=1}^{n} a_i = -1.$$

We can obtain the solution for $(n+1)$-dimensional equation (2) by substituting (18) into (16)

$$u_0 = -\frac{ck\phi_{1'}(\xi)}{\phi_1(\xi)}, \quad u_i = \frac{k\phi_{1'}(\xi)}{\phi_1(\xi)}, \quad i = 1, 2, \cdots, n. \tag{19}$$

Due to the solution (19) including an arbitrary function ϕ, we can obtain different forms of solution with selection ϕ. We shall select the arbitrary function ϕ to be Jacobian elliptic and hyperbolic functions as the explicit example. The motivation behind this choice of arbitrary function stems from the fact that the limiting forms of these functions happen to be localized functions [18]. Here, we take (2+1)-dimensional extension system as example and choose the arbitrary function ϕ as

$$\phi = \mathrm{dn}(\xi, m), \tag{20}$$

where k and c are arbitrary constants, m is the modulus of the Jacobi elliptic function. We can obtain the solution u_1 of the (2+1)-dimensional system

$$u_1 = \frac{km^2 \mathrm{cn}(\xi, m)\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}, \tag{21}$$

In the left panel of Fig. 1, the solution u_1 of (21) with the parameters $k = c = 1$ and $m = 0.1$ Furthermore, the solution u_1 is obtained

$$u_1 = \frac{\tanh(\xi)\mathrm{dn}(\xi, \mathrm{m}) + m^2 \mathrm{cn}(\xi, \mathrm{m})\mathrm{sn}(\xi, \mathrm{m}) + \mathrm{dn}(\xi, \mathrm{m})}{\mathrm{dn}(\xi, \mathrm{m})}, \tag{22}$$

by selecting

$$\phi = \mathrm{dn}(\xi, \mathrm{m}) - \tanh(\xi)\mathrm{dn}(\xi, \mathrm{m}). \tag{23}$$

The corresponding solution u_1 of (22) is plotted with the parameters $k = c = 1$ and $m = 0.7$ in the right panel of Fig. 1. Solution (22) describes a kind of periodic-kink interaction solitary wave which has been obtained [19]. The left panel of Fig. 2 shows the exact solution u_1 by selecting

$$\phi = \cosh(\xi) - \mathrm{cn}(\xi, m), \tag{24}$$

and choosing the parameters $k = c = 1$ and $m = 0.1$. While the field ϕ reads as

$$\phi = 1 - \mathrm{sech}(\xi) + \mathrm{sech}(\xi)^2, \tag{25}$$

the solution u_1 is shown with the parameters $k = c = 1$ in the right panel of Fig. 2.

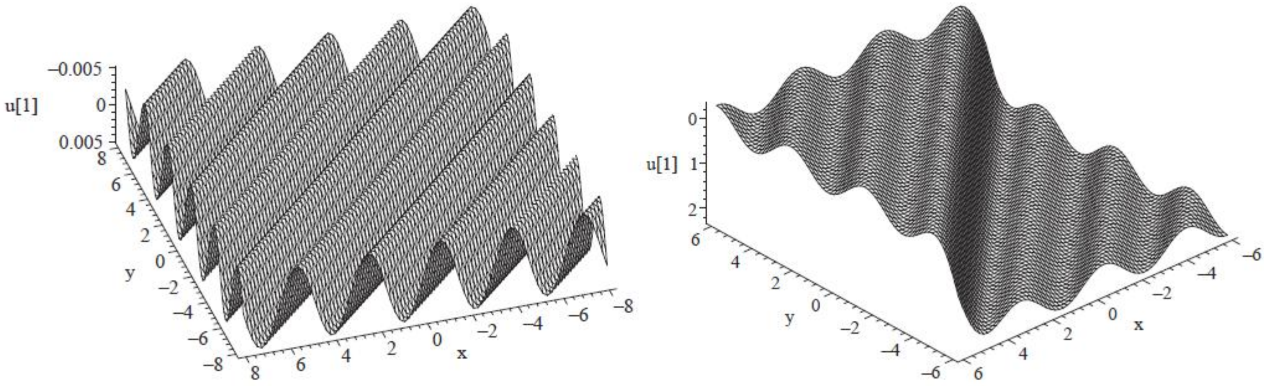

Figure 1: Evolution of the solution u_1 (21) and (22) at $t = 0$, respectively.

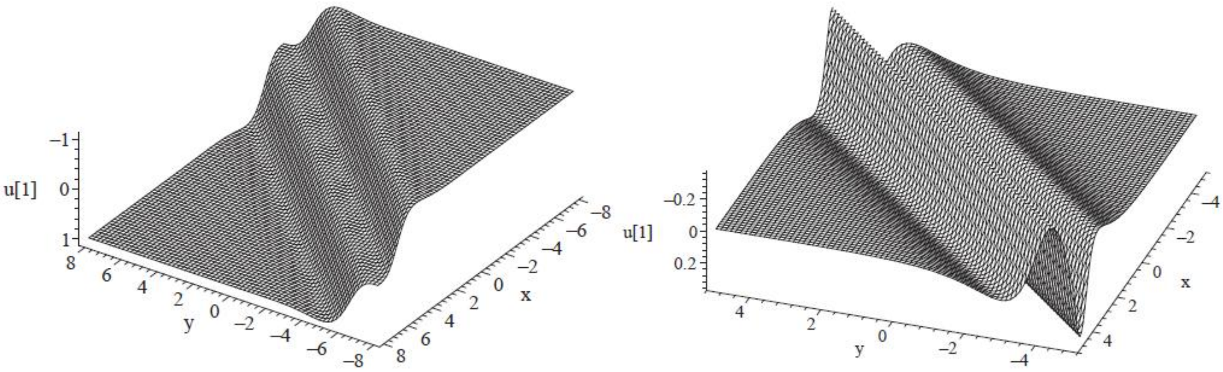

Figure 2: With selecting ϕ as (24) and (25), the similar kink solution and the soliton solution u_1 at $t = 0$, respectively.

Conclusion

In summary, we have extended the (1+1)-dimensional Schwartz CDG equation to the arbitrary dimensional system. With the standard WTC method, we have shown that the new system satisfies the Painlevé property and invariant under the Möbius transformation. By the Painlevé-Bäcklund transform, the traveling wave solutions are obtained for the $(n+1)$-dimensional system. Meanwhile, the properties of the soliton solutions for the extension system are shown by some figures. More properties of the $(n+1)$-dimensional integrable system such as multi-soliton solutions, infinitely many conservation laws and symmetries are worthy further studying.

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