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Warped Product Semi Slant Submanifold of Nearly Quasi Sasakian Manifold

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Abstract

The main objective of this paper is to study some geometric properties of warped product semi-slant submanifold of nearly quasi sasakian manifold.

Keywords: Warped product, slant submanifold, Semi-slant submanifold, Nearly quasi sasakian structure.

1. Introduction

The notion of slant submanifolds of an almost contact metric manifold were introduced by J. L. Cabrerizo et al. [8]. The study of semi-slant submanifolds of almost Hermitian manifolds was initiated by N. Papaghuic [17]. In fact, semi-slant submanifolds in almost Hermitian manifolds are defined on the line of CR-submanifolds. Cabrerizo et al. [7] introduced almost contact metric manifolds, semi-slant submanifolds and studied their properties.

Also, the notion of warped products manifolds with negative curvature was defined and studied by Bishop and O'Neill [6]. B.Y. Chen [9] extended the work the work of Bishop and O'Neill and studied the warped product CR-submanifold of Kachler manifolds and many mores and was followed up by several other authors.

In [3], Blair introduced the notion of Quasi Sasakian structure. Since then several papers on quasi-Sasakian manifolds have studied by Tanno [27], Kanemaki [11, 12], Oubina [16], and the author and et al., [18-24]. Kim [14] extensively studied quasi-Sasakian manifolds and proved that fibred Riemannian spaces with invariant fibrs normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosympletic structure.

In the present paper we study warped product semi-slant submanifold of nearly quasi sasakian manifold. The paper is divided into three sections. In Section 2 we recall some necessary detail of a nearly quasi sasakian manifold. In Section 3, we prove that the warped product semi-slant submanifolds of the type $M = N_\theta \times_f N_T$ do not exist in a nearly quasi Sasakian manifold $\bar{M}$. However, we obtain some interesting results on the existence of the warped product submanifolds of the type $M = N_T \times_f N_\theta$ of a nearly quasi Sasakian manifold $\bar{M}$, where N_T and N_θ are the invariant and proper slant submanifolds of $\bar{M}$, respectively.

2. Preliminaries

Suppose $\bar{M}$ be a real $(2n+1)$ dimensional differentiable manifold endowed with an almost contact metric structure (f, ξ, n, g), where f is a tensor field of type $(1,1)$, vector field , η is a 1-form and g is a Riemannian metric on $\bar{M}$ s. t.

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta o \phi = 0, \quad \phi(\xi) = 0, \quad \eta(X) = g(X, \xi)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. An Almost contact metric structure (f, ξ, η, g) on $\bar{M}$ is called Quasi Sasakian manifold if

$$(\bar{\nabla}_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.2}$$

where A is symmetric lineartransformationfield , $\bar{\nabla}$ denotes the Riemannian connection of on $\bar{M}$. If in a addition to above relations

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi \tag{2.3}$$

then $\bar{M}$ is called a nearly quasi –Sasakian manifold. We have also on nearly quasi Sasakian manifold $\bar{M}$,

$$\bar{\nabla}_X \xi = \phi AX$$

Suppose M be a submanifold of $\bar{M}$. Then the induced Riemannian metric on M is denoted by the same symbol g and the induced Riemannian connection by ∇. Further, if $T\bar{M}$ and TM denote the tangent bundle on $\bar{M}$ and on M respectively and $T^\perp M$, the normal bundle on M, then the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + \hbar(X, Y) \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{2.5}$$

for each $X, Y \in TM$ and $N \in T^\perp M$, h and A_N denote respectively the second fundamental forms and the shape operator (corresponding to the normal vector field N) of the immersion of M into M. The two are related as:

$$g(\hbar(X, Y), N) = g(A_N X, Y) \tag{2.6}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as induced on M.

Further, for any $p \in M$, let $\{e_{1}, e_2, e_3, \dots \dots . e_m, \dots \dots e_{2n+1}\}$ be an orthogonal frame for the tangent space $T_p\bar{M}$ such that $e_1, e_2, \dots \dots \dots, e_m$ are tangent to M at . We denote by H the mean curvature vector, that is

$$H(p)= \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$$

Also we get,

$$h_{ij}^{r} = g(h(e_i, e_i), e_r), \quad i,j \in \{1,2, \ldots\ldots, m\}, \quad r \in \{m+1, m+2, \ldots\ldots 2n+1\}$$

and $\|h\|^2 = \sum_{i=1}^{m} g(h(e_i, e_j), h(e_i, e_j))$.

For any $X \in \Gamma(TM)$, we write,

$$\phi X = PX + FX \qquad (2.7)$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly for any $N \in \Gamma(T^{\perp}M)$, we write

$$\phi N = BN + CN \qquad (2.8)$$

where BN is the tangential component and CN is the normal component of ϕN.

The covariant derivative of the tensor field ϕ is defined as

$$(\overline{\nabla}_X \phi)Y = \overline{\nabla}_X \phi Y - \phi \overline{\nabla}_X Y \qquad (2.9)$$

for all $X, Y \in \Gamma(T\overline{M})$.

Now, denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\overline{\nabla}_X \phi)Y$, i.e.,

$$(\overline{\nabla}_X \phi)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \qquad (2.10)$$

for all $X, Y \in \Gamma(TM)$. Making use of (2.7)-(2.10) and the Gauss and Weingarten formulae, the following equations may easily be obtained

$$\mathcal{P}_X Y = (\overline{\nabla}_X P)Y - A_{FY}Y - Bh(X, Y) \qquad (2.11)$$

$$\mathcal{Q}_X Y = (\overline{\nabla}_X F)Y + h(X, PY) - Ch(X, Y) \qquad (2.12)$$

where the covariant derivative of P and F are defined by

$$(\overline{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y \qquad (2.13)$$

$$(\overline{\nabla}_X F)Y = \nabla_X^{\perp} FY - F\nabla_X Y \qquad (2.14)$$

for all $X, Y \in \Gamma(TM)$.

Similarly, for any $X \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, denoting the tangential and normal parts of $(\overline{\nabla}_X \phi)N$ by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$ respectively, we obtain

$$\mathcal{P}_X N = (\overline{\nabla}_X B)N + PA_N X - A_{CN} X \qquad (2.15)$$

$$\mathcal{Q}_X N = (\overline{\nabla}_X C)N + h(BN, X) + FA_N X \qquad (2.16)$$

where the covariant derivative of B and C are defined by

$$(\overline{\nabla}_X B)N = \nabla_X BN - B\nabla_X^{\perp} N \qquad (2.17)$$

$$(\overline{\nabla}_X C)N = \nabla_X^{\perp} CN - C\nabla_X^{\perp} N \qquad (2.18)$$

It is uncomplicated to verify the following properties of P and Q, which we enrol here for later on use:

(p_1) (i) $\mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W$ (ii) $\mathcal{Q}_{X+Y}W = \mathcal{Q}_X W + \mathcal{Q}_Y W$

(p_2) (i) $\mathcal{P}_X(Y+W) = \mathcal{P}_X Y + \mathcal{P}_X W$ (ii) $\mathcal{Q}_X(Y+W) = \mathcal{Q}_X Y + \mathcal{Q}_X W$

(p_3) (i) $g(\mathcal{P}_X Y, W) = -g(Y, \mathcal{P}_X W)$ (ii) $g(\mathcal{Q}_X Y, N) = -g(Y, \mathcal{Q}_X N)$

for all $X, Y, W \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$.

On a submanifold M of a nearly quasi Sasakian manifold, by equations (2.3) and (2.10) we have

(a) $\mathcal{P}_X Y + \mathcal{P}_Y X = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi$, (b) $\mathcal{Q}_X Y + \mathcal{Q}_Y X = 0$ (2.19)

for all $X, Y \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be invariant if F is identically zero, that is, $\phi X \in \Gamma(TM)$ and anti-invariant if P is identically zero, that is, $\phi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.

We shall always consider ξ to be tangent to the submanifold M. There is another class of submanifolds called slant. For each non zero vector X tangent to M at x, such that X is not proportional to ξ_x, we denote by $0 \leq \theta(X) \leq \pi/2$, the angle between ϕX and $T_x M$ is called the slant angle. If the slant angle $\theta(X)$ is constant for all $X \in T_x M - \langle \xi_x \rangle$ and $x \in M$, then M is said to be slant submanifold [8]. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant submanifold.

We recall the following result for a slant submanifold.

Theorem 2.1 [8] If M be an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ such that $\xi \in (TM)$, then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that

$$P^2 = \lambda(-I + \eta \otimes \xi) \tag{2.20}$$

Also, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Next relations are basic consequences of relation (2.20)

Corollary 2.2 If M be a slant submanifold of a Lorentzian almost paracontact manifold $(\bar{M}, \phi, \xi, \eta, g)$ with $\xi \in \Gamma(TM)$, then

$$g(PX, PY) = cos^2\theta\big(g(X,Y) - \eta(Y)\eta(X)\big) \tag{2.21}$$

$$g(FX, FY) = sin^2\theta\big(g(X,Y) - \eta(Y)\eta(X)\big) \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be semi-slant if there exist two orthogonal complementary distributions D_1 and D_2 satisfying:

(i) $TM = D_1 \oplus D_2 \oplus \langle \xi \rangle$

(ii) D_1 is an invariant i.e., $\phi D_1 \subseteq D_1$

(iii) D_2 is a slant distribution with slant angle $\theta \neq \frac{\pi}{2}$.

A semi-slant submanifold M of an almost contact manifold $\bar{M}$ is mixed geodesic if

$$\hbar(X,Z) = 0 \tag{2.23}$$

for any $X \in D_1$ and $Z \in D_2$. Moreover, if μ is the $\phi-$invariant subspace of the normal bundle $T^{\perp}M$, then in case of semi-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = FD_2 \oplus \mu \tag{2.24}$$

3. Warped product semi-slant submanifolds

Suppose (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and $f > 0$ a differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1: N_1 \times N_2 \to N_1$ and $\pi_2: N_1 \times N_2 \to N_2$. Then the warped product of N_1 and N_2 denoted by $M = N_1 \times_f N_2$ is a Riemannian manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X,Y) = g_1(\pi_{1*}X, \pi_{1*}X) + (f \circ \pi)^2 g_1(\pi_{2*}X, \pi_{2*}X)$$

for each $X, Y \in \Gamma(TM)$ and $*$ is a symbol for the tangent map. Thus we have

$$g = g_1 + f^2 g_2 \tag{3.1}$$

The function f is called the warping function of the warped product [2, 11]. A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function f is constant. We recall the following general result for a warped product manifold for later use.

Lemma 3.1 [6] If $M = N_1 \times_f N_2$ be a warped product manifold, then

(i) $\nabla_X Y \in TN_1$ is the lift of $\nabla_X Y$ on N_1.

(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$

(iii) $\nabla_Z W = \nabla_Z^{N_2} W - g(Z,W)\nabla \ln f$

for all $X, Y \in \Gamma(TN_1)$ and $Z, W \in \Gamma(TN_2)$ where ∇ and ∇^{N_2} denote the Levi-Civita connections on M and N_2, respectively, and $\nabla \ln f$ is the gradient of $\ln f$.

Suppose M be a Riemannian manifold of dimension k with the inner product g and $\{e_1, \dots, e_k\}$ be an orthonormal frame on M. Then for a differentiable function f on M, the gradient ∇f of a function f on M is defined by

$$g(\nabla f, U) = Uf \tag{3.2}$$

for any $U \in \Gamma(TM)$. As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{k} \big(e_i(f)\big)^2 \tag{3.3}$$

where ∇f is the gradient of the function f on M.

Now, we consider warped product semi-slant submanifolds tangent to the structure vector field ξ which are either in the form $M = N_T \times_f N_\theta$ or $M = N_\theta \times_f N_T$ in a nearly quasi Sasakian manifold $\bar{M}$, where N_T and N_θ are invariant and proper slant submanifolds of a nearly quasi

Sasakian manifold $\bar{M}$, respectively. On a warped product submanifold $M = N_1 \times_f N_2$ of a nearly quasi Sasakian manifold $\bar{M}$, we have the following result.

Theorem 3.1. Suppose $M = N_1 \times_f N_2$ be a warped product semi-slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$. Then M is usual Riemannian product if the structure vector field ξ is tangent to N_2, where N_1 and N_2 are the Riemannian submanifolds of $\bar{M}$.

Starting the above theorem for the existence of warped products we always consider the structure vector field ξ is tangent to the base. First we discuss the warped product semi-slant submanifolds of the type $M = N_\theta \times_f N_T$ of a nearly quasi Sasakian manifold $\bar{M}$.

Theorem 3.2 If $\bar{M}$ be a nearly quasi Sasakian manifold, then there do not exist warped product semi-slant submanifolds $M = N_\theta \times_f N_T$ such that N_θ is a proper slant submanifolds and N_T is invariant submanifolds $\bar{M}$.

Now, we will discuss the other case

Lemma 3.2 If $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$ such that N_T and N_θ are invariant and proper slant submanifolds of $\bar{M}$, then

(i) $\xi \ln f = 0$

(ii) $g(\hbar(X,Y), FZ) = 0$

(iii) $g(\hbar(PX,Z), FZ) = (X\, lnf)\|Z\|^2$

(iv) $g(h(X,Z), FPZ) = -\, g(h(X,PZ), FZ)$

(v) $g(\mathcal{P}_X Z, PZ) = 2g(h(X,Z), FPZ)$

for all $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Proof. Consider $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$. We assume that the structure vector field ξ is tangent to N_T, then for any $Z \in \Gamma(TN_\perp)$, we have

$$(\bar{\nabla}_\xi \phi)Z + (\bar{\nabla}_Z \phi)\xi = AZ$$

Using (2.4), we obtain

$$-\bar{\nabla}_\xi \phi Z + \phi(\bar{\nabla}_\xi Z) + \phi(\bar{\nabla}_Z \xi) = -AZ$$

Then from Lemma 2.1(ii) and (2.5), we derive

$$-\,\bar{\nabla}_\xi \phi Z + 2\phi\hbar(Z,\xi) + 2\phi(\xi\, lnf)Z \ = -AZ$$

Taking inner product with ϕZ, we have

$$(\xi \ln f)\|Z\|^2 \ = 0 \Rightarrow \ \xi(lnf) = 0$$

This means that either M is invariant or $\xi \ln f = 0$, which proves (i). Now, we consider $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$, the we have

$$g(h(X,Y), FZ) = g(\overline{\nabla}_X Y, FZ) = -g(Y, \overline{\nabla}_X FZ)$$

with (2.7) and (2.9), we obtain

$$g(h(X,Y), FZ) = -g(Y, (\overline{\nabla}_X \phi)Z) - g(Y, \phi\overline{\nabla}_X Z) + g(Y, \overline{\nabla}_X PZ)$$

Then from (2.2), (2.4) and Lemma 3.1 (ii), the second and last terms of right hand side vanish identically and hence by (2.10), we derive

$$g(h(X,Y), FZ) = -g(Y, \mathcal{P}_X Z)$$

Thus, on using the property p_3 (i), we get

$$g(h(X,Y), FZ) = g(\mathcal{P}_X Y, Z)$$

Hence, by skew-symmetry of $\mathcal{P}_X Y$ and symmetry of $h(X,Y)$, we get the second part of the lemma.

For the third part, consider for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$, we have

$$g(h(PX,Y), FZ) = g(\overline{\nabla}_Z PX, \phi Z - PZ)$$
$$= -g(PX, \overline{\nabla}_Z \phi Z) - g(\overline{\nabla}_Z PX, PZ)$$

From (2.4), (2.9) and Lemma 3.1 (ii), the above equation is reduced to

$$g(h(PX,Y), FZ) = -g(PX, (\overline{\nabla}_Z \phi)Z) - g(PX, \phi\, \overline{\nabla}_Z Z) - PX \ln f\, g(Z, PZ)$$

On using the structure equation of a nearly quasi Sasakian manifold and the fact that Z and PZ are orthogonal vector fields, the first and last terms of the right hand side are identically zero. Then from (2.2) we derive

$$g(h(PX,Y), FZ) = g(\phi^2 X, \overline{\nabla}_Z Z)$$

Using (2.1), we get

$$g(h(PX,Y), FZ) = -g(X, \overline{\nabla}_Z Z) + \eta(X) g(\xi, \overline{\nabla}_Z Z)$$

By the property of Riemannian connection $\overline{\nabla}$, the above equation takes the form

$$g(h(PX,Y), FZ) = g(\overline{\nabla}_Z Z, X) - \eta(X) g(\overline{\nabla}_Z \xi, Z)$$

Then from (2.4), Proposition 2.1 and Lemma 3.1 (ii), we obtain

$$g(h(PX,Y), FZ) = (X \ln f)\|Z\|^2$$

which is third part of the lemma. For the other parts, consider

$$g(\nabla_{PZ} \phi X, Z) = g(\overline{\nabla}_{PZ} \phi X, Z)$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$. Using the property of Riemannian connection $\overline{\nabla}$ and Lemma 3.1 (ii), we get

$$(\phi X \ln f) g(Z, PZ) = -g(\phi X, \overline{\nabla}_{PZ} Z)$$

Using the fact that Z and PZ are orthogonal vector fields, the above equation reduces to

$$g(X, \phi\overline{\nabla}_{PZ} Z) = 0$$

Then form (2.9), we derive

$$0 = g(X, \overline{\nabla}_{PZ}\phi Z) - g(X, (\overline{\nabla}_{PZ}\phi)Z)$$

By (2.7) and (2.10), we obtain

$$0 = g(X, \overline{\nabla}_{PZ}PZ) + g(X, \overline{\nabla}_{PZ}FZ) - g(X, \mathcal{P}_{PZ}Z)$$

Using (2.4), (2.5) and (2.19) (a), we get

$$0 = -g(\nabla_{PZ}X, PZ) - g(X, A_{FZ}PZ) + g(X, \mathcal{P}_Z PZ)$$

Then from the property p_3 (i) and Lemma 3.1 (ii), we obtain

$$0 = -(X \ln f)g(PZ, PZ) - g(h(X, PZ), FZ) - g(\mathcal{P}_Z X, PZ)$$

Again with (2.19) (a), (2.21) and the fact that ξ is tangent to N_T, we derive

$$g(\mathcal{P}_X Z, PZ) = (X\, lnf)cos^2\theta\|Z\|^2 + g(h(X, PZ), FZ) \tag{3.4}$$

Interchanging Z by PZ and then using (2.20), (2.21) and the fact that ξ is tangent to N_T, we obtain

$$-cos^2\theta\, g(\mathcal{P}_X PZ, Z) = (X \ln f)\, cos^4\theta\, \|Z\|^2 - cos^2\theta\, g(h(X, Z), FPZ)$$

By the property p_3 (i), the above equation will be

$$(\mathcal{P}_X Z, PZ) = (X \ln f)\, cos^2\theta\, \|Z\|^2 - g(h(X, Z), FPZ) \tag{3.5}$$

Thus, the fourth part of the lemma follows from (3.4) and (3.5). Now, for the part (v), we consider

$$g(\hbar(X, PY), FZ) = g(\overline{\nabla}_X PZ, FZ)$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$. Using the property of Riemannian connection $\overline{\nabla}$ and then using (2.7), we have

$$g(\hbar(X, PY), FZ) = -g(PZ, \overline{\nabla}_X \phi Z) + g(PZ, \overline{\nabla}_X PZ)$$

Using (2.9), Lemma 3.1 (ii), (2.21) and the fact that ξ is tangent to N_T, we obtain

$$g(\hbar(X, PY), FZ) = -g(PZ, \phi\overline{\nabla}_X Z) - g(PZ, (\overline{\nabla}_X\phi)Z) + (X \ln f)cos^2\theta\, \|Z\|^2$$

Then from (2.2) and (2.10), we get

$$g(\hbar(X, PY), FZ) = g(\phi PZ, \overline{\nabla}_X Z) - g(PZ, \mathcal{P}_X Z) + (X \ln f)cos^2\theta\, \|Z\|^2$$

Using (2.4) and (2.7), we derive

$$g(\hbar(X, PY), FZ) = g(P^2 Z, \nabla_X Z) + g(h(X, Z), FPZ)$$
$$-g(PZ, \mathcal{P}_X Z) + (X\, lnf)cos^2\theta\|Z\|^2$$

Again, from the fact that ξ is tangent to N_T and using (2.20), the above equation reduces to

$$g(\mathcal{P}_X Z, PZ) = g(h(X, Z), FPZ) - g(h(X, PZ), FZ) \tag{3.6}$$

Thus, the fifth part of the lemma follows from (3.6) and the fourth part of this lemma. This proves the lemma completely.

Theorem 3.3 If $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold in a nearly quasi Sasakian manifold $\bar{M}$ such that N_T and N_θ are invariant and proper slant submanifolds of $\bar{M}$, then

$$g(\mathcal{P}_X Z, PZ) = \frac{2}{3}(X \ln f)cos^2\theta\|Z\|^2$$

for all $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Proof. With (3.4) and (3.5), we obtain

$$2g(\mathcal{P}_X Z, PZ) = 2(X \ln f)cos^2\theta\|Z\|^2 + g(h(X, PZ), FZ) - g(h(X, Z), FPZ) \qquad (3.7)$$

Then, by Lemma 3.2 (iv), we obtain

$$2g(\mathcal{P}_X Z, PZ) = 2(X \ln f)cos^2\theta\|Z\|^2 - 2g(h(X, Z), FPZ) \qquad (3.8)$$

Thus, from Lemma 3.2 (v) and (3.8), we obtain the desire result.

From the above theorem we have the following consequence.

Corollary 3.1 If $M = N_T \times_f N_\theta$ is a warped product semi-slant submanifold of a nearly quasi Sasakian manifold $\bar{M}$ is a Riemannian product of N_T and N_θ if and only if $\mathcal{P}_X Z \in \Gamma(TN_T)$, for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Theorem 3.4 If $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold in a nearly quasi Sasakian manifold $\bar{M}$ such that N_T and N_θ are invariant and proper slant submanifolds of $\bar{M}$, respectively, then

$$g(h(X, Z), FPZ) = -g(h(X, PZ), FZ) = \frac{1}{3}(X\, lnf)cos^2\theta\|Z\|^2$$

for all $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Proof. The first equality is nothing but Lemma 3.2 (iv) and the second equality is directly followed by the equation (3.8) and Lemma 3.2 (v).

From the above theorem we have the following corollaries.

Corollary 3.2 Suppose $M = N_T \times_f N_\theta$ be a semi-slant warped product submanifold of a nearly quasi Sasakian manifold $\bar{M}$. Then M is simply a Riemannian product of N_T and N_θ if and only if $h(X, Z) \in \Gamma(\mu)$, for all $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$, where μ is the invariant normal subbundle of $T^\perp M$.

Corollary 3.3 If $\bar{M}$ be a nearly quasi Sasakian manifold, then there do not exist a mixed geodesic warped product semi-slant submanifold $M = N_T \times_f N_\theta$ such that N_T is invariant submanifolds and N_θ is proper slant submanifolds $\bar{M}$.

From Lemma 3.2 (i), (iv) and Theorem 3.4, we obtain

$$g(h(\xi, Z), FPZ) = g(h(\xi, PZ), FZ) = 0, \qquad (3.9)$$

for any $Z \in \Gamma(TN_\theta)$.

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