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Inhomogeneous Density and Energy Spectrum of Elementary Excitations of Electron Liquid of Conductors in Magnetic Field.

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Abstract

It is shown that the conventional theory of electron liquid in a magnetic field is based on the model that is incorrect mathematically and contradicting to experiments. On the foundation of statistical mechanics that takes into account angular momentum conservation law is obtained the effective Hamiltonian of the electron liquid in a magnetic field that includes the quasielectrostatic potential proportional to the square of the magnetic field. As result, the electron liquid is inhomogeneous, and together with the homogeneous neutralizing background generates an electrostatic potential. The sum of these potentials is the residual potential that forms an energetically spectrum of elementary excitations of the electron liquid in the homogeneous magnetic field. This spectrum is quasicontinuous, and its beginning is somewhat higher than the value of the potential energy at the boundary. The graph of the dependence of the density of states on energy has the form of a staircase with horizontal steps. The density of states in the mean is proportional to the square root from energy.

Keywords: Statistical operator; effective Hamiltonian; electron liquid; energy functional; inhomogeneous density; residual potential; elementary excitation; density of states.

1. Introduction

The celebrated Landau formula for magnetic susceptibility (see Ref. [1])

$$\chi_{\mathrm{dm}} = -\frac{(3n_{el})^{1/3} e^2}{12\pi^{4/3} m_e}, \quad n_{el} = \frac{N}{V} \tag{1}$$

cannot be correct. This formula describes the quantum effect, but it does not satisfy the requirement of the correspondence principle, since it does not depend on the Planck constant. In any stationary state of an electron in a magnetic field there is a closed electric current. It generates a magnetic moment. The averaged magnetization field can be homogeneous only if all diamagnetic currents are closed in an atomic volume. But for an electron in the uniform magnetic field the operators of the classical orbital center coordinates do not commute. That is why the streamlines can cover regions with a diameter comparable with the dimensions of the volume under consideration. Therefore, a specific susceptibility makes no sense. It is obvious that a disk perpendicular to the magnetic field and a thin cylinder whose axis is parallel to this field must have different magnetic moments though their volumes are equal.

A magnetic field can be homogeneous only on an arbitrary, but confined part of a plane because a magnetic field is vortex field, and hence its lines must be closed curves. Therefore the problem of an electron in a magnetic field must be considered at the condition that wave function equals to zero on the boundary of the homogeneous magnetic field. There are two solutions of this task those are reported in the monograph [2], but the both solutions are obtained with incorrect boundary condition. The Landau solution described in the main body of the text is incorrect also because it does not correspond to the axial symmetry of this task (for more details, see Ref. [3]). In the solution by Johnson and Lippmann (see Ref. [4]), which is printed in the Ref. [2] as a task with solution, this mistake was corrected. In both solutions the

plane was assumed to be unbounded, and the eigen functions to be normalizable. The results about energy spectrum in the both solutions are identical and do not have the physical sense. The spectrum is discrete and equidistant for any energy, and all eigenvalues are infinitely degenerated, because they are independent from quantum number of angular momentum, if it is negative. That is result of the consideration of an unbounded plane. For using in statistical thermodynamics degeneracy order of all eigenvalues are arbitrarily limited by value that depended from the magnetic field. The formula (1) is deduced from this spectrum. If the electrons were to fill states that are described by the usual theory, then the angular momentum of the gas would be large negative.

The global properties of spectrums of like tasks on a bounded plane are known (see to monograph [5]). A proper consideration of the task about an electron in the uniform magnetic field was impossible because in mathematical books (see, for example, monograph [6]) the largest zero of the degenerate hypergeometric function $\Phi(a,c;z)$ with $a=-n-\gamma$, and $0<\gamma<1$ is not described. As we have proved in the work [7] this zero tends to infinity when $\gamma\to 0$, and it tends to the largest zero of the Laguerre polynomial of degree $n+1$ when $\gamma\to 1$. Therefore this zero must be utilized for satisfaction of the boundary condition in this task. In the work [7] we have proved also that the eigenvalues of the energy are little varied with altering of the quantum number of the angular momentum, but the amount of the permissible values of this number is bounded. That determinates the quasi-degeneration of the eigenvalues. When energies of eigenstates are large, the spectrum is quasi-continuous and the state density is constant. This spectrum satisfies the requirements that are proved by mathematics (see to monograph [5]). Specifically, for example, the minimal eigenvalue is non-degenerate.

Although the commonly known statistical thermodynamics of the electron gas in the magnetic field proceeds from unfounded assumptions and incorrect energy spectrum, it explains many experimental effects (see to monographs [8, 9]). These results are consequence of the discreteness of the assumed spectrum.

V. L. Ginzburg called the construction of the theory of a two-dimensional electron liquid, which is a system of electrons with taking into account of the Coulomb interaction, among the thirty most important problems of physics (see Ref. [10]). The properties of the electron system in a magnetic field are mainly determined by the properties of the two-dimensional subsystem in a plane perpendicular to the magnetic field. This theory was developed in Refs. [7, 11 – 14]. The present work is the continuation of them.

In the conventional theory it is suggested that the electrons in a uniform magnetic field homogeneously fill the area that is filled by the homogeneous positive electrical charge. This suggestion contradicts to experimental observations. It is known that the uniform magnetic field restrains a gas of charged particles from expansion. In the equilibrium state the density of the electron liquid in the uniform magnetic field must be inhomogeneous. This inhomogeneity will generate the electrostatic field that can explain some phenomena in the grate of conductor. These phenomena cannot be explained by the theory that does not take into account the non-homogeneous density of the electron liquid. It is necessary in the theory of the electron liquid explain this inhomogeneity. This explaining necessitates changes in the density of distribution of the classical statistical ensemble and the statistical operator of the quantum statistics. Those changes described in the works [7, 11 - 14].

These changes are founded on the idea by A. Y. Khinchin (see to monographs [15, 16]) that the density of distribution and the statistical operator must depend from everybody motion integrals of the considered system. The inclusion of the conservation of zero value of the angular momentum that is parallel to the magnetic field into the statistical operator substantially changes results. In the works [7, 11, 13] and in the second section of this work it is proved that this conservation lead to the effective Hamiltonian of the electron liquid. This Hamiltonian does not contain terms linear with respect to the magnetic field. The term of the Hamiltonian quadratic with respect to the magnetic field has the form of the potential energy of the electron liquid in a parabolic potential. The shielding of this potential is considered in the third section. The inhomogeneity of the electron density generates a potential that reduces this energy. This shielding cannot reduce this energy to zero. The residual potential determines the spectrum of elementary excitations of the electron liquid. It will be considered in the fourth section. The full electrostatic potential effects on immovable charges of the background. The fifth section is the mathematical supplement in which is considered the theorem about Macdonald function that is used in the fourth section.

2.The statistical operator for the electron liquid in the uniform magnetic field and description of model

The statistical operator $\hat{\mathcal{P}}\left(\hat{\mathcal{H}};E\right)$ for the microcanonical distribution with the Hamiltonian $\hat{\mathcal{H}}$ and energy E can be expressed through the characteristic operator $\hat{\varphi}\left(\hat{\mathcal{H}};E\right)$:

$$\hat{\varphi}\left(\hat{\mathcal{H}};E\right)=\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\left(i\tau+\vartheta\right)\left(\hat{\mathcal{H}}-E\right)\right]\mathrm{d}\tau\,. \tag{2}$$

Here ϑ is an arbitrary real number that will be determined, if we shall pass to the statistical operator for a system in a thermostat (see Ref. [13]). This operator in the space of eigenfunctions of the Hamiltonian $\hat{\mathcal{H}}$ has only diagonal matrix elements, which are equal toone only if they are obtained with eigenfunctions belonging to the subspace of eigenvalue E .The other matrix elements of this operator are zero. The statistical operator is $\hat{\mathcal{P}}\left(\hat{\mathcal{H}};E\right)=\hat{\varphi}\left(\hat{\mathcal{H}};E\right)\left[\mathrm{Sp}\left\{\hat{\varphi}\left(\hat{\mathcal{H}};E\right)\right\}\right]^{-1}$.

A. Y. Khinchin has pointed that the statistical operator of the equilibrium system must be determined in the space of eigenfunctions not only of the Hamiltonian but also other the integrals of motion that commute with the Hamiltonian and with each other. The characteristic operator for the subspace in which the energy has the value E , and the component of the angular momentum $\hat{\mathcal{L}}_z$ has the value L_z is

$$\hat{\varphi}\left(\hat{\mathcal{H}},\hat{\mathcal{L}}_z;E,L_z\right)=\left[\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\left(i\tau+\vartheta\right)\left(\hat{\mathcal{H}}-E\right)\right]\mathrm{d}\tau\right]\left[\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\left(i\alpha+\lambda\right)\left(\hat{\mathcal{L}}_z-L_z\right)\right]\mathrm{d}\alpha\right]. \tag{3}$$

The Hamiltonian of the electron liquid in the uniform magnetic field is:

$$\hat{\mathcal{H}}=\sum_{i=1}^{N}\hat{\mathfrak{H}}_i+U\left(\{\mathbf{r}_i\}\right);\quad \hat{\mathfrak{H}}_i=\frac{1}{2m}\hat{\mathbf{p}}_i^2+\frac{e^2B^2}{8m}\left(x_i^2+y_i^2\right)+\frac{eB}{2m}\hat{l}_{z,i};\qquad \hat{l}_{z,i}=x_i\hat{p}_{y,i}-y_i\hat{p}_{x,i}\,, \tag{4}$$

where N is the number of electrons, m is the mass and $(-e)$ is the charge of an electron, $\hat{\mathbf{p}}_i=\left(p_{x,i},p_{y,i},p_{z,i}\right)$ is the its momentum, $\mathbf{r}_i=\left(x_i,y_i,z_i\right)$ is the radius-vector in the configuration space, $U\left(\{\mathbf{r}_i\}\right)$ is the potential energy of the Coulomb interaction between electrons and between electrons and a homogeneous neutralizing background, $\{\mathbf{r}_i\}$ are radius-vectors of all electrons, B is the induction of the magnetic field. We neglect the electron spin energy in the magnetic field, and shall consider all one-particle states as doubly degenerated. We substitute (4) into (3), taking into account that $\sum_{i=1}^{N}\hat{l}_{z,i}=\hat{\mathcal{L}}_z$, and put $L_z=0$. Then we obtain:

$$\begin{aligned}\hat{\varphi}\left(\hat{\mathcal{H}},\hat{\mathcal{L}}_z;E,L_z\right)&=\left[\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\left(i\alpha+\lambda\right)\hat{\mathcal{L}}_z\right]\mathrm{d}\alpha\right]\times\\&\times\left[\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\left(i\tau+\vartheta\right)\left(\sum_{i=1}^{N}\left\{\frac{1}{2m}\hat{\mathbf{p}}_i^2+\frac{e^2B^2}{8m}\left(x_i^2+y_i^2\right)\right\}+U\left(\{\mathbf{r}_i\}\right)-E\right)\right]\mathrm{d}\tau\right].\end{aligned} \tag{5}$$

It is evident that the statistical operator in this case (when $L_z=0$) is determined by the Hamiltonian of a liquid of charged particles in the quasielectrostatic parabolic field $M\left(\rho\right)=-\left(eB^2/8m\right)\left(x^2+y^2\right)$ that we will name as M-potential. This potential differs from electrostatic potential, since it must not satisfy the continuity condition. The density of the liquid will be heterogeneous for a shielding this potential. The full compensating is impossible, as it usually is. If the M-potential and the field of the electron density heterogeneity in full mutually cancel out to one another, this density distribution will be destructed by diffusion.

A conductor under consideration must be bounded. We shall suppose that it is a circular cylinder of radius R and height L . The axis of cylinder is directed parallel to the magnetic field and is the axis $\mathbf{Z}$. In what follows $\mathbf{r}=\left(\rho,\varphi,z\right)$, and all functions are axially symmetric.

An inhomogeneous electron liquid with a homogeneous positive charged background creates an uncompensated charge. Potential that generated by this uncompensated charge satisfies the requirement of continuity and therefore must be nonzero and depend on z beyond of the cylinder. We may assume that the density of the uncompensated charge don't depend on z inside the cylinder. The number of electron per unit volume of the crystal is $\sigma_0=jk/a^3$, where k is the number of ions in the cell of the crystal lattice, ej is the ion charge, and a is the crystal lattice constant. We shall consider the problem about the magnetic field influence with help the method of the theory of the inhomogeneous electron gas (see to Ref. [17]). It is necessary to introduce changes in this theory taking into account the symmetry of the

M-potential. Only the $\rho-$ component of the momentum shall be dependent on the coordinates under the effect of this potential. We denote this component as $p_{\perp}$, and the z-component will be denoted as $p_{\parallel}$.The main supposing of this theory in our case is described by the equation

$$\frac{p_{\perp F}^2(\rho)}{2m} - ew_0(\rho) = \zeta \ , \tag{6}$$

where $p_{\perp F}(\rho)$ is the value of the $p_{\perp}$ on the Fermi level when $p_{\parallel}=0$. It depends on ρ because the density of electrons is inhomogeneous. The function $w_0(\rho)$ is a sum of the M-potential with the shielding potential $W_0(\rho)$, that must be get as the task solution. A new energy Fermi ζ does not depend on ρ. We represent $p_{\perp F}^2(\rho)$ in the form $p_{\perp F}^2(\rho) = p_F^2\left[1+\alpha\chi(\rho)\right]$, where $p_F = \sqrt{2m\zeta_0}$ is a radius of a Fermi surface and $\zeta_0 = \left(\hbar^2/2m\right)\left(3\pi^2\sigma_0\right)^{2/3}$ is the Fermi energy, when the magnetic field is absent. Here $\alpha = B^2\varepsilon_0/2m\sigma_0$ (ε_0 is electric constant) is a ratio of the Laplacian of the M-potential to the Laplacian of the potential that generated by the background $\alpha = B^2\cdot 5\cdot 10^{-11}$. It can be considered as the small parameter of the theory. The function $\chi(\rho)$ must be obtained as the result of the solution of the problem. The Fermi surface for an element of a volume that have the coordinate ρ is an ellipsoid of revolution. The transverse semi-axis of this ellipsoid is $p_F\sqrt{1+\alpha\chi(\rho)}$, and the density of states in this ellipsoid is equal to $(8/3)(2\pi\hbar)^{-3}\, p_F^3\left[1+\alpha\chi(\rho)\right]$. The density of states is equal to the density of electrons. Consequently, the density of electrons in an homogeneous magnetic field is $\sigma(\rho) = \sigma_0\left[1+\alpha\chi(\rho)\right]$. The potential that is generated the inhomogeneity of the electron density must have the view:

$$W_0(\rho) = -M(\rho) + w_0(\rho) = \left(eB^2/8m\right)\rho^2 + w_0(\rho). \tag{7}$$

The first it's term $-M(\rho)$ compensates the quasielectrostatic field that effects electron liquid only. It also produces the principal part of the magnetic field effect on any other charges. For example, it can generate a diffusion of charged impurity atoms. The residual potential $w_0(\rho)$ is significantly less than $M(\rho)$. This potential determines a spectrum of elementary excitations of an electron liquid in the homogeneous magnetic field.

3. The equations for the density of electron liquid and for electrostatic potential

We shall deduce these equations by the method of a density functional (see to Ref. [17]). The energy of the inhomogeneous electron liquid is composed from the kinetic energy of electrons, the potential energy of the liquid in the field of M-potential, and from the potential energy of self-action of the uncompensated charge.

The kinetic energy of an electron in an element of volume of a space of momenta, which has the coordinates $(p_{\perp}, p_{\parallel})$, equals to $\left(p_{\perp}^2 + p_{\parallel}^2\right)/2m$. The probability that the state is in this element is $3p_{\perp}\mathrm{d}p_{\perp}\mathrm{d}p_{\parallel}\mathrm{d}\varphi/4\pi p_F^3\left(1+\alpha\chi\right)$. The momentum component $p_{\perp}$ in the volume element of the coordinate space can have values $0 \le p_{\perp} \le p_F\sqrt{1+\alpha\chi(\rho)}$. Then the modulus of $p_{\parallel}$ can have values $0 \le |p_{\parallel}| \le p_{\parallel\max}$,where $p_{z(\max)} = \sqrt{\left|p_F^2 - p_{\perp}^2\left[1+\alpha\chi(\rho)\right]^{-1}\right|}$. The average kinetic energy of an electron in the volume element of the coordinate space $2\pi\rho\mathrm{d}\rho\mathrm{d}z$ is

$$\overline{t(\rho)} = \frac{3}{2m\left(2m\zeta_0\right)^{3/2}\left[1+\alpha\chi(\rho)\right]}\int_0^{p_{\perp F}} p_{\perp}\mathrm{d}p_{\perp}\int_0^{p_{z(\max)}}\left(p_{\perp}^2+p_z^2\right)\mathrm{d}p_z = \frac{2}{5}\zeta_0\left[1+\alpha\chi(\rho)\right] + \frac{\zeta_0}{5}. \tag{8}$$

The number of electrons in this volume element is $2\pi\sigma_0\left[1+\alpha\chi(\rho)\right]\rho\mathrm{d}\rho\mathrm{d}z$. Then the full kinetic energy of the electron liquid is

$$T = 2\pi L\sigma_0\int_0^R(1+\alpha\chi)\overline{t(\rho)}\rho\mathrm{d}\rho = \frac{2\pi L\sigma_0\zeta_0}{5}\int_0^R\left[3+5\alpha\chi+2\alpha^2\chi^2\right]\rho\mathrm{d}\rho = \frac{3\pi R^2L\zeta_0}{5}\sigma_0 + 2\pi\alpha L\zeta_0\sigma_0\int_0^R\chi(\rho)\rho\mathrm{d}\rho + \frac{4\pi\alpha^2L\zeta_0}{5}\sigma_0\int_0^R\chi^2(\rho)\rho\mathrm{d}\rho \ . \tag{9}$$

The density of the uncompensated charge is $q=-e\sigma_0\alpha\chi(\rho)$. The energy of self-action of the uncompensated charge is

$$E_{s-a}\left[\chi(\rho)\right]=\frac{1}{2\varepsilon_0}\iint\limits_{VV'}q(\mathbf{r})G(|\mathbf{r}-\mathbf{r}'|)q(\mathbf{r}')\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}', \tag{10}$$

where $G(|\mathbf{r}-\mathbf{r}'|)=\left\{4\pi\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z-z')}\right\}^{-1}$ is the Green function of the Poisson equation. Then the energy of the fundamental state of the electron liquid in the magnetic field is

$$\begin{aligned}E\left[\chi(\rho)\right]=&\frac{3\pi R^2L\zeta_0\sigma_0}{5}+2\pi L\alpha\zeta_0\sigma_0\int\limits_0^R\chi(\rho)\rho\mathrm{d}\rho+\frac{4\pi L\alpha^2\zeta_0\sigma_0}{5}\int\limits_0^R\chi^2(\rho)\rho\mathrm{d}\rho+\\&\frac{1}{2\varepsilon_0}\iint\limits_{VV'}q(\mathbf{r})G(|\mathbf{r}-\mathbf{r}'|)q(\mathbf{r}')\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'+\frac{2\pi Le^2B^2\sigma_0}{8m}\int\limits_0^R\left[1+\alpha\chi(\rho)\right]\rho^3\mathrm{d}\rho\end{aligned}. \tag{11}$$

The functional $E\left[\chi(\rho)\right]$ should be minimized under the following supplementary condition:

$$\int\limits_0^R\chi(\rho)\rho\mathrm{d}\rho=0. \tag{12}$$

We use the Lagrange method: multiply the additional condition (12) by a constant $(-2\pi L\alpha\zeta\sigma_0)$, where ζ is the factor that must determine the new Fermi energy (see formula (6)). We add this product to the functional (11). By equalization of the first-order variation to zero we obtain the equation for determination of the function $\chi(\rho)$:

$$2\pi L\sigma_0\alpha(\zeta_0-\zeta)\rho+\frac{8\pi L\sigma_0\alpha^2\zeta_0}{5}\chi(\rho)\rho-\frac{e\sigma_0\alpha\rho}{\varepsilon_0}\int\limits_{-L/2}^{L/2}\int\limits_0^{2\pi}\mathrm{d}z\mathrm{d}\varphi\int\limits_{V'}G(|\mathbf{r}-\mathbf{r}'|)q(\mathbf{r}')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'+\frac{2\pi L\sigma_0\alpha e^2B^2}{8m}\rho^3=0. \tag{13}$$

Denote the potential that is generated by uncompensated charge $W(\rho,z)$:

$$W(\rho,z)=\frac{1}{\varepsilon_0}\int\limits_{V'}G(|\mathbf{r}-\mathbf{r}'|)q(\mathbf{r}')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'=\frac{2\pi}{\varepsilon_0}\int\limits_{-\infty}^{\infty}\int\limits_0^R G(|\mathbf{r}-\mathbf{r}'|)q(\mathbf{r}')\xi(z'/L)\rho'\mathrm{d}\rho'\mathrm{d}z', \tag{14}$$

where $\xi(z/L)=\left[\Theta(z/L+1/2)-\Theta(z/L-1/2)\right]$, and $\Theta(x)=\begin{cases}1 \text{ when } x>0,\\0 \text{ when } x<0\end{cases}$. It is the solution of the Poisson equation

$$\Delta W(\rho,z)=-q(\rho)\xi(z/L)/\varepsilon_0. \tag{15}$$

This potential does not depend on φ. When $-L/2<z<L/2$, potential does not depend on z also, and therefore in this limits the equation (15) is reduced to the view:

$$\begin{aligned}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)W(\rho)=\Delta_\rho W(\rho)=-\frac{q(\rho)}{\varepsilon_0}=\frac{e\sigma_0\alpha}{\varepsilon_0}\chi(\rho)\\\chi(\rho)=\frac{\varepsilon_0}{e\sigma_0\alpha}\Delta_\rho W(\rho)\end{aligned}. \tag{16}$$

We shall transform the integral

$$\frac{1}{\varepsilon_0}\int\limits_{-L/2}^{L/2}\int\limits_0^{2\pi}\mathrm{d}z\mathrm{d}\varphi\int\limits_{V'}G(|\mathbf{r}-\mathbf{r}'|)q(\mathbf{r}')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z'\equiv 2\pi\int\limits_{-L/2}^{L/2}\xi(z/L)W(\rho,z)\mathrm{d}z=2\pi LW(\rho). \tag{17}$$

The formula (13) with using of this potential has the view:

$$(\zeta_0-\zeta)+\frac{4\zeta_0}{5}\frac{\varepsilon_0}{e\sigma_0}\Delta_\rho W(\rho)-eW(\rho)+\frac{e^2B^2}{8m}\rho^2=0. \tag{18}$$

This potential $W(\rho)$ must be potential $W_0(\rho)$, which was presumed in the formula (7). Substituting (7) into (18) get:

$$(\zeta_0-\zeta)+\frac{2\zeta_0}{5}\frac{\varepsilon_0}{\sigma_0}\frac{B^2}{m}+\frac{4\zeta_0}{5}\frac{\varepsilon_0}{e\sigma_0}\Delta_\rho w_0-ew_0=0. \tag{19}$$

Averaging the equation (6) over area we obtain:

$$\zeta-\zeta_0=-\frac{2e}{R^2}\int_0^R w_0(\rho)\rho\mathrm{d}\rho\ . \tag{20}$$

Substituting (20) into (19), we obtain the equation for the function $w_0(\rho)$:

$$\Delta_\rho w_0-\Lambda^2 w_0+\Lambda^2\frac{2}{R^2}\int_0^R w_0(\rho)\rho\mathrm{d}\rho=-\frac{eB^2}{2m},\ \text{where}\ \Lambda^2=\frac{5e^2\sigma_0}{4\zeta_0\varepsilon_0}=10^{20}\,\mathrm{m}^{-2}\ , \tag{21}$$

when $\sigma_0=4\cdot10^{29}\,\mathrm{m}^{-3}$. The solution of this equation that equals to zero when $\rho=0$:

$$w_0(\rho)=-\left[\mathbf{I}_1(\Lambda R)\Lambda R\right]^{-1}\frac{eB^2R^2}{4m}\left[\mathbf{I}_0(\Lambda\rho)-1\right], \tag{22}$$

where $\mathbf{I}_k(t)$ is a modified Bessel function. The parameter Λ^{-1} is the characteristic length of a shielding. It is easily shown that solution of the similar problem for a gas of classical charged particles leads to the similar equation (see to monograph [18]), in which Λ^{-1} is the Debye-Hückel screening radius. The quantum theory of shielding in [18] leads to $\Lambda^2=3e^2\sigma_0/2\zeta_0\varepsilon_0$. The difference in the numerical coefficients is explained by the axial symmetry of this problem.

4. Elementary excitations energy spectrum in the residual potential $w_0(\rho)$

The residual potential is very small because of the great value ΛR in the denominator of the formula (22). But we will show that due to the nature of its increase, which is close to exponential, it creates a zone of elementary excitations whose density of states has a special form that can to explain magnetic oscillations.

From axial symmetry it follows that the Hamiltonian can be presented as $\hat{\mathfrak{K}}(\rho,\varphi)+\hat{\mathfrak{K}}(z)$, and first we consider eigenstates only for $\hat{\mathfrak{K}}(\rho,\varphi)$:

$$\hat{\mathfrak{K}}(\rho,\varphi)=-\frac{\hbar^2}{2m*}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)+(-e)w_0(\rho), \tag{23}$$

where m^* is an effective mass of elementary excitation. The boundary condition for the relevant Schrödinger equation $\hat{\mathfrak{K}}(\rho,\varphi)\psi(\rho,\varphi)=E_\perp\psi(\rho,\varphi)$ is $\psi(R,\varphi)=0$. Because $\Lambda R>>1$ we can consider the Schrödinger equation with asymptotic view of the potential energy:

$$\begin{gathered}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{l^2}{\rho^2}\right)\psi_l(\rho)+\left\{\tilde{E}-G\frac{\exp(\Lambda\rho)}{\sqrt{2\pi\Lambda\rho}}\right\}\psi_l(\rho)=0,\\ \frac{2m^*}{\hbar^2}E=\tilde{E},\quad \frac{m^*e^2B^2R}{2\hbar^2m\Lambda\left[\mathrm{I}_1(\Lambda R)\right]}=G,\end{gathered} \tag{24}$$

where l is a quantum number of the angular momentum. We introduce a change of the variable:

$$\begin{gathered}u=\mathbf{exp}\left(\frac{\Lambda\rho}{2}\right),\ \ \rho=\frac{2}{\Lambda}\mathbf{ln}(u)\\ \left[\frac{\partial^2}{\partial u^2}+\frac{1}{u}\left(\frac{\partial}{\partial u}+\frac{1}{\mathbf{ln}\,u}\frac{\partial}{\partial u}\right)-\frac{l^2}{u^2(\mathbf{ln}\,u)^2}+\frac{4\tilde{E}}{\Lambda^2u^2}-\frac{4G}{\Lambda^2\sqrt{4\pi\,\mathbf{ln}\,u}}\right]\psi_l(u)=0\end{gathered}\ . \tag{25}$$

In the neighborhood of the bound $\mathbf{ln}\,u=\Lambda\rho/2>>1$, and we can suppose $\mathbf{ln}\,u\approx\Lambda R/2>>1$. Then this equation has the asymptotic view:

$$\left[\frac{\partial^2}{\partial^2u}+\frac{1}{u}\frac{\partial}{\partial u}+\frac{4\left(\tilde{E}-l^2/R^2\right)}{\Lambda^2u^2}-\frac{2m*e^2B^2R}{\hbar^2m\Lambda^3\left[\mathbf{exp}(\Lambda R)\right]}\right]\psi_l(u)=0\,. \tag{26}$$

The only solution to this equation that has zeros is

$$\psi(u)=\mathbf{K}_{i\tau}(\gamma u),\quad \gamma=\frac{eB}{\Lambda\hbar}\sqrt{\frac{2Rm^*}{m\Lambda}}\,\mathbf{exp}\left(-\frac{\Lambda R}{2}\right),\quad \tau=\frac{2}{\Lambda}\sqrt{\tilde{E}-l^2/R^2}\,. \tag{27}$$

Here $\mathbf{K}_{i\tau}(t)$ $(t=\gamma u)$ is the Macdonald function with imaginary index. The boundary value of the argument, when $\rho=R$: $t_b=sB$, where $s=e\sqrt{2Rm^*/m\Lambda^3\hbar^2}\approx 0{,}86\left(m^*/m\right)^{1/2}$ T^{-1}. (For numerical estimates we take $R=0.16$ m). We shall prove in the mathematical supplement (see to the fifth section of this work) that when $\mathbf{K}_{i\tau}(t_b)=0$ the relation between the argument and indexes is:

$$t_b\left[-\sqrt{\theta^2-1}+\theta\ln\left(\theta+\sqrt{\theta^2-1}\right)\right]=p_n,\quad p_n=\pi\left(n+\frac{1}{2}\right),\quad \theta=\frac{\tau_n}{t_b}>1, \tag{28}$$

where n is arbitrary integral number or zero. It has been known (see the monograph [19]) that $\tau_0>t_b$, $\theta>1$. This equation determines the infinite increasing sequence $\{\tau_n\}$ of the index values. From the formula (27) it follow that the every value τ_n determines the origin of an one-dimensional zone of quasi-discrete values of the energy:

$$E(n,l)=\frac{\hbar^2\Lambda^2}{8m^*}\tau_n^2+\frac{\hbar^2l^2}{2m^*R^2}\,. \tag{29}$$

Hamiltonian $\hat{\mathfrak{K}}(z)$ describes an one-dimensional free motion. The energy spectrum of this motion is $E(k)=\left(\hbar^2/2m\right)\left(\pi k/L\right)^2$. A convolution of densities of states of two one-dimensional zones leads to a density of a two-dimensional zone that asymptotically is constant. The energy of an elementary excitation is determined by three quantum numbers: radial - n, angular momentum - l, and free motion that is parallel to the magnetic field - k:

$$E(n,l,k)=\frac{\hbar^2\Lambda^2}{8m^*}\tau_n^2+\frac{\hbar^2l^2}{2m^*R^2}+\frac{\hbar^2k^2}{2mL^2}\,. \tag{30}$$

The density of states in an energy interval $\hbar^2\Lambda^2\tau_n^2/8m^*<E<\hbar^2\Lambda^2\tau_{n+1}^2/8m^*$ with taking into account the spin degeneration is equal to

$$\mathfrak{N}(E)=\frac{\sqrt{mm^*}RL}{\pi\hbar^2}\sum_{i=0}^{n}\int_0^{E-E_i}\frac{\mathrm{d}v}{\sqrt{v\left(E-E_i-v\right)}}=\frac{\sqrt{mm^*}RL}{\hbar^2}(n+1)\,. \tag{31}$$

In this formula the asymptotical view of the density of states of a two-dimensional zone is extrapolated to the zero point as usual. (As it was seen in the work Ref. [14], when the eigenvalues of the energy in one of the two one-dimensional zones are close to zero the energy spectrum can't be described by the density of states.) Therefore, the graph of the dependence of the density of states on energy for the three-dimensional system under consideration has the form of a staircase with horizontal steps. The positions of the bounds of step on the staircase are determined by the values of τ_n^2:

$$E(n,0,0)=\frac{\hbar^2\Lambda^2}{8m^*}\tau_n^2=\frac{\hbar^2\Lambda^2}{8m^*}t_b^2\theta_n^2=\frac{\hbar^2\Lambda^2}{8m^*}\frac{2e^2B^2R}{\Lambda^3\hbar^2}\frac{m^*}{m}\theta_n^2=\frac{e^2B^2R}{4m\Lambda}\theta_n^2\,. \tag{32}$$

The energy $e^2B^2R/4m\Lambda$ is equal to the energy of an electron on the boundary, where the potential $w_0(\rho)$ has the maximal value (see the formula (22)). Hence the quasicontinuous spectrum begins above this value because $\theta_0^2>1$. To determine the positions of the steps, a solution of equation (28) is required. Let us transform this equation by expansion in terms of θ^{-2}:

$$\left[-\sqrt{\theta^2-1}+\theta\ln\left(\theta+\sqrt{\theta^2-1}\right)\right]=\theta\left[\ln\left(\frac{2\theta}{\mathrm{e}}\right)+\frac{1}{4\theta^2}+O\left(\theta^{-4}\right)\right]=\frac{p_n}{t_b}. \tag{33}$$

Here e is the Napierian base. The first-order correction must be taken into account because the solution of the equation with the expansion dominant term only has the view $\theta_n=\left(p_n/t_b\right)f\left[\ln\left(p_n/t_b\right)\right]$. Therefore an index $\tau_n=t_b\theta_n$ will depend from t_b only logariphmically. The solution with taking into account the first-order correction can be obtained by iteration. Let us simplify the equation with the dominant term:

$$\frac{2\theta}{\mathrm{e}}\ln\left(\frac{2\theta}{\mathrm{e}}\right)=\mu\ln\mu=\frac{2p_n}{\mathrm{e}t_b}=\frac{2\pi}{\mathrm{e}}\left(n+\frac{1}{2}\right)\frac{\hbar\Lambda}{eB}\sqrt{\frac{\Lambda m}{2Rm^*}}=\frac{1{,}07}{B}\sqrt{\frac{m}{Rm^*}}\left(n+\frac{1}{2}\right)=g_n\,. \tag{34}$$

As a solution of the equation $\mu\ln\mu=g_n$, when $g_n>\mathrm{e}$, can be accepted $\mu_n=g_n/\ln\left(g_n\right)$. In so doing the relative error in maximum is 6% and vanish when $g_n\to\infty$. Then

$$\theta_n^{(0)} = \frac{p_n}{t_b}\left[\ln\left(\frac{2p_n}{\mathrm{e}t_b}\right)\right]^{-1};\quad \theta_n \ln\left(\frac{2}{\mathrm{e}}\theta_n\right) = q_n;\quad \theta_n = q_n\left[\ln\left(\frac{2q_n}{\mathrm{e}}\right)\right]^{-1},$$
$$\text{where}\quad q_n = \frac{p_n}{t_b} - \frac{t_b}{4p_n}\left[\ln\left(\frac{2p_n}{\mathrm{e}t_b}\right)\right] \qquad (35)$$

This formulae can be used in neighborhood of the Fermi energy, where n are large. In this region the positions of the steps are determined by $(t_b\theta_n)^2$:

$$E(n,0,0) = \frac{\hbar^2\Lambda^2}{8m^*}(t_b\theta_n)^2 \approx \frac{\hbar^2\Lambda^2}{8m^*}\left\{p_n^2 - \frac{t_b^2}{2}\left[\ln\left(\frac{2p_n}{\mathrm{e}t_b}\right)\right]\right\}\left[\ln\left(\frac{2q_n}{\mathrm{e}}\right)\right]^{-2}. \qquad (36)$$

Here the first term in the braces mainly determines the step position. From the formulae (31) and (36) follows that the density of states increases, on average, proportional to $\sqrt{E}$, when the energy is large, as it must be. This term do not depend on the magnetic field. The second term is smaller, but it is proportional to B^2. The logariphmically multipliers also depend on the magnetic field. When the magnetic field increases, the positions of the steps are shifted towards lower energies.

In order that bottom edge of band can be not far from the maximum of the potential energy (see to formula (32)) $\theta_0^2 - 1$ must be small. Let $\theta_0^2 - 1 = \gamma^2 < 1$, and substitute this formula into equation (28). We obtain:

$$\left[-\sqrt{\theta^2-1} + \theta\ln\left(\theta + \sqrt{\theta^2-1}\right)\right] \approx \frac{\gamma^3}{3} = \frac{\pi}{2t_b};\quad \gamma^2 \simeq 1{,}7\left(B^2R\right)^{-1/3}\left(m/m^*\right)^{1/3}. \qquad (37)$$

For $\gamma^2 < 1$ must be $(m/m^*) \cong 10^{-3}$. The energy ζ (see to formula (20)) is measured from the bottom edge of band. The step number n_F, on which the Fermi level lies, is determined by approximate equation:

$$E_F \approx \frac{\pi^2\hbar^2\Lambda^2}{8m^*}\left(\frac{n_F}{\ln q_F}\right)^2 \approx \frac{e^2B^2R}{4m\Lambda} + \frac{\hbar^2}{2m}\left(3\pi^2\sigma_0\right)^{2/3}. \qquad (38)$$

When the magnetic field increases, the number n_F will increase by jump. This jump occurs when the decreasing energy of the next step (see Formula (36)) is equal to the Fermi level. These jumps can explainmagnetic oscillations.

5. The zeroes of the Macdonald function with imaginary index

The Macdonald function, which has an imaginary index, can be represented by the integral (see Ref [20]):

$$K_{i\tau}(t) = \int_0^\infty \exp\left[-t\cosh(z)\right]\cos(\tau z)\,\mathrm{d}z =$$
$$\frac{1}{2}\left\{\int_0^\infty \exp\left[-t\cosh(z)\right]\exp(i\tau z)\,\mathrm{d}z + \int_0^\infty \exp\left[-t\cosh(z)\right]\exp(-i\tau z)\,\mathrm{d}z\right\}. \qquad (39)$$

Let us consider the first integral. We shall transform it to a contour integral, and the contour will be selected just as in the method of steepest descent (see Ref [20]):

$$\int_0^\infty \exp\left[-t\cosh(z) + i\tau z\right]\mathrm{d}z \to \int_S \exp\left(-t\cosh\left[x+iy\right] + i\tau\left[x+iy\right]\right)\mathrm{d}s. \qquad (40)$$

We have to find the stationary point of function $f(z) = \left(-t\cosh(z) + i\tau z\right)$, i.e. we have to solve equation $-t\sinh\left[x+iy\right] + i\tau = 0$. We denote in the stationary point $x = \alpha,\ y = \beta$ and introduce $\theta = \tau/t > 1$ (see Ref [19]). Then

$$-\sinh(\alpha)\cos(\beta) - i\cosh(\alpha)\sin(\beta) + i\theta = 0$$
$$\begin{cases}\sinh(\alpha)\cos(\beta) = 0 \\ -\cosh(\alpha)\sin(\beta) + \theta = 0\end{cases}\quad\begin{cases}\beta = \pi/2 \\ \alpha = \mathrm{Arccosh}(\theta) = \ln\left(\theta + \sqrt{\theta^2-1}\right)\end{cases}. \qquad (41)$$

The contour is chosen so that it passes through the stationary point (40). We may expect that an approximate value of the integral will be determined from a consideration of the integrand in the neighborhood of the stationary point. Then the contour line of steepest descent in this neighborhood is determined by the assumption that the imaginary part of the exponent is constant on it and is equal to its value in the stationary point:

$$-t\sinh(\alpha)+\tau\alpha = t\left[-\sqrt{\theta^2-1}+\theta\ln\left(\theta+\sqrt{\theta^2-1}\right)\right]. \tag{42}$$

In this approximation (saddle point or steepest descent approximation) the phase factor can be taken out from under integral, and we obtain:

$$\begin{aligned}&\int_S \exp\left(-t\cosh\left[x+iy\right]+i\tau\left[x+iy\right]\right)\mathrm{d}s \approx \\ &\exp\left\{it\left[-\sqrt{\theta^2-1}+\theta\ln\left(\theta+\sqrt{\theta^2-1}\right)\right]\right\}\int_S \exp\left\{-t\left[\cosh(x)\cos(y)+\theta y\right]\right\}\mathrm{d}s\end{aligned}. \tag{43}$$

The contour S originates in the point ($x=0$, $y=0$). It passes through stationary point as discussed above and when $x\to\infty$ asymptotically converges to the straight line $y=0$. The second integral in the formula (39) is complex conjugate to the first integral, and we obtain:

$$K_{i\tau}(t) \approx \cos\left\{t\left[-\sqrt{\theta^2-1}+\theta\ln\left(\theta+\sqrt{\theta^2-1}\right)\right]\right\}\int_S \exp\left\{-t\left[\cosh(x)\cos(y)-\theta y\right]\right\}\mathrm{d}s\,. \tag{44}$$

If we fix the argument $t=t_b$, then indexes of all functions $K_{i\tau}(t)$ that have zero when $t=t_b$ shall be determined by the equation:

$$t_b\left[-\sqrt{\theta^2-1}+\theta\ln\left(\theta+\sqrt{\theta^2-1}\right)\right] = \pi\left(n+\frac{1}{2}\right). \tag{45}$$

6. Conclusion

It is shown that the usual theory of an electron gas in a homogeneous magnetic field is based on incorrect assumptions and leads to some results that contradict experiments. A new theory of an electron liquid in a homogeneous magnetic field, which is created on the basis of statistical mechanics taking into account the law of conservation of angular momentum, has been suggested. In this theory, the density of the electron liquid is inhomogeneous and with a uniform background creates an electric field. The graph of the dependence of the density of states of elementary excitations of an electron liquid on energy has the form of a staircase with horizontal steps. The density of states on average increases in proportion to the square root of the energy.

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